Definition 1. A set \( A \) in \( \mathbb{R}^n \) is convex if whenever \( x \) and \( y \) are in \( A \), then
\[
(1 - t)x + ty
\]
is contained in \( A \) for all \( t, 0 \leq t \leq 1 \).

That is, if \( x \) and \( y \) are in \( A \), then the line segment connecting them is contained in \( A \).

Problem 1 (Intersection of two convex sets). If \( A \) and \( B \) are convex sets, then \( A \cap B \) is convex.

Problem 2 (Intersection of arbitrarily many convex sets). Let \( S \) be a non-empty set. For each \( s \in S \), let \( A_s \) be a convex subset of \( \mathbb{R}^n \). Then \( \bigcap_{s \in S} A_s \) is convex. [Note that \( \bigcap_{s \in S} A_s := \{ x \in \mathbb{R}^n \mid x \in A_s \text{ for all } s \in S \} \].

Definition 2. Let \( A \) be a set in \( \mathbb{R}^n \). The convex hull of \( A \), denoted \( C_H(A) \), is the intersection of all convex sets containing \( A \). [Note that the convex hull is convex by Problem 2 and that \( A \) is contained in the convex hull, since \( A \) is contained in at least one convex set, namely \( \mathbb{R}^n \).]

Note 3. The convex hull is the smallest convex set containing \( A \), that is, if \( B \) is a convex set containing \( A \) then the convex hull of \( A \) is contained in \( B \).

Problem 3. Let \( A \) and \( B \) be sets in \( \mathbb{R}^n \), \( A \subseteq B \). Then \( C_H(A) \subset C_H(B) \).

Definition 4. Let \( A := \{v_1, ..., v_m\} \) be a finite set in \( \mathbb{R}^n \). We denote by \( \mathcal{C}(A) \) the set of all convex combinations of \( \{v_1, ..., v_m\} \), that is, \( \mathcal{C}(A) := \{ p_1v_1 + ... p_mv_m \mid p_i \geq 0 \text{ for all } i \text{ and } \sum p_i = 1 \} \).

Proposition 5. Let \( A := \{v_1, ..., v_m\} \) be a finite set in \( \mathbb{R}^n \). Then \( \mathcal{C}(A) \) is convex.

Proof. Let \( p_1v_1 + ... p_mv_m \) and \( q_1v_1 + ... q_mv_m \) be in \( \mathcal{C}(A) \), where \( p_i \geq 0 \) for all \( i \) and \( \sum p_i = 1 \) and \( q_i \geq 0 \) for all \( i \) and \( \sum q_i = 1 \). We have to show that if \( 0 \leq t \leq 1 \), then
\[
(1 - t)(p_1v_1 + ... p_mv_m) + t(q_1v_1 + ... q_mv_m)
\]
is in \( \mathcal{C}(A) \). But \( (1 - t)(p_1v_1 + ... p_mv_m) + t(q_1v_1 + ... q_mv_m) = ((1 - t)p_1 + t_q1)v_1 + ... + ((1 - t)p_m + t_qm)v_m \). This is in \( \mathcal{C}(A) \) since \( (1 - t)p_i + t_qi \geq 0 \) for all \( i \) (since \( (1 - t), p_i, t, q_i \) are all \( \geq 0 \)), and since \( \sum (1 - t)p_i + t_qi = (1 - t)\sum p_i + t\sum q_i = (1 - t) \cdot 1 + t \cdot 1 = 1 - t + t = 1 \).

Proposition 6. Let \( A := \{v_1, ..., v_m\} \) be a finite set in \( \mathbb{R}^n \). If \( B \) is a convex set containing \( A \), then \( B \) contains \( \mathcal{C}(A) \).

Proof. Let \( B \) be a convex set containing \( A \). We show by induction on \( m \) (the number of elements in \( A \)) that \( B \) contains \( \mathcal{C}(a) \).

True for \( m = 1 \): If \( A = \{v_1\}A \), then \( v_1 \in B \), since \( B \) contains \( A \). Easy to check that \( \mathcal{C}(A) = \{v_1\} \) in this case.

Assume true for \( m = k \). Then for \( m = k + 1 \), let \( p_i \geq 0 \) and \( \sum p_i = 1 \). To show that \( p_1v_1 + ... p_{k+1}v_{k+1} \) is in \( B \):
Let \( p_{k+1} = t \), so \( 0 \leq t \leq 1 \) (why is \( t \leq 1 \)?). If \( t = 1 \) then \( p_1 = ... = p_k = 0 \) (why?), so \( p_1 v_1 + ... + p_{k+1} v_{k+1} \) is just \( v_{k+1} \in A \), so in \( B \). So assume \( 0 \leq t < 1 \). Let \( q_i = p_i / (1 - t) \) for \( i = 1, \ldots, k \). Then \( q_i \geq 0 \) and \( q_1 + ... + q_k = (p_1 + ... + p_k) / (1 - t) = (1 - p_{k+1}) / (1 - t) = 1 \). So \( q_1 v_1 + ... + q_k v_k \) is in \( B \) by assumption and 
\[
p_1 v_1 + ... + p_{k+1} v_{k+1} = (1 - t)(q_1 v_1 + ... + q_k v_k) + tv_{k+1}\]
is in \( B \) by convexity. \( \square \)

**Proposition 7.** Let \( A := \{v_1, ..., v_m\} \) be a finite set in \( \mathbb{R}^n \). Then \( C_H(A) = C(A) \), that is, the set of all convex combinations of a finite set is the convex hull of the set.

**Proof.** We know that \( C(A) \) contains \( A \), is convex, and is contained in every convex set \( B \) containing \( A \). To show \( C(A) \) is the convex hull of \( A \):

The convex hull is the intersection of all convex sets \( B \) containing \( A \). Since \( C(A) \) is contained in each convex set \( B \) containing \( A \) by the previous proposition, then \( C(A) \) is contained in the intersection of all convex sets \( B \) containing \( A \), so \( C(A) \) is contained in the convex hull.

Now let \( B_0 = C(A) \). Then \( B_0 \) is a convex set containing \( A \). So the intersection of all convex sets \( B \) containing \( A \) is contained in any one of the convex sets, in particular is contained in \( B_0 \). So the convex hull is contained in \( B_0 = C(A) \). \( \square \)