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BOXES OF COMPACT ORDINALS

by

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If $\{X_n: n \in \omega\}$ is a family of spaces, then $\square_{n\in\omega}X_n$, called the box product of those spaces, denotes the cartesian product of the sets with the topology generated by all sets of the form $\Pi_{n\in\omega}G_n$, where each G_n need only be open in the factor space X_n. If $X_n = X$ $\forall$ $n\in\omega$, we denote $\square_{n\in\omega}X_n$ by $\square^{\omega}X$.

M. E. Rudin [5] and K. Kunen [3 and 6, pg. 58] have shown that CH implies $\square_{n\in\omega}(\lambda_n + 1)$ is paracompact for every countable collection of ordinals $\{\lambda_n: n \in \omega\}$. At the 1976 Auburn University Topology Conference I demonstrated [7] that the paracompactness of $\square^{\omega}(\omega + 1)$ is implied by the existence of a k-scale in ${}^{\omega}\omega$, a set-theoretic axiom which is a consequence of, but not equivalent to, Martin's Axiom, and hence CH. In addition, I proved $\square^{\omega}(\omega_1 + 1)$ is paracompact iff $\square^{\omega}(\alpha + 1)$ is paracompact $\forall$ countable ordinals α. If this is coupled with E. van Douwen's ($\exists$ a k-scale in ${}^{\omega}\omega$) $\Rightarrow$ ${}_{n\in\omega}X_n$ is paracompact for all collections $\{X_n: n \in \omega\}$ of compact metrizable spaces [1], we have $\square^{\omega}(\omega_1 + 1)$ is paracompact if $\exists$ a k-scale in ${}^{\omega}\omega$. However, none of the proofs generalize to higher ordinals ($\square^{\omega}(\omega_2 + 1)$, for example). We conjecture:

If $\square^{\omega}(\omega + 1)$ *is paracompact, then* $\square^{\omega}(\lambda + 1)$ *is paracompact* $\forall$ *ordinals* λ.[1]

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[1]It is unknown whether it is consistent for $\square^{\omega}(\omega + 1)$ not to be paracompact; however, $\exists$ compact spaces X_n such that $\square_{n\in\omega}X_n$ is not normal. Moreover, irrationals $\times(\square^{\omega}(\omega + 1))$ is not normal [6, pg. 58].

Toward this conjecture we show:

Suppose λ is an ordinal for which $\sqcup_{n\in\omega}(\lambda_n + 1)$ is paracompact whenever $\lambda_n < \lambda \, \forall \, n \in \omega$, then $\sqcup^{\omega}(\lambda + 1)$ if either of the following holds:

(1) $cf(\lambda) \neq \omega$ (Theorem 1).

(2) $cf(\lambda) = \omega$ and $\exists$ a k-scale in ${}^{\omega}\omega$ (Theorem 2).

Now suppose $\{X_n: n \in \omega\}$ is a family of sets and for each $f \in \Pi_{n\in\omega}X_n$,

$$E(f) = \{g \in \Pi_{n\in\omega}X_n: (\exists \, m \in \omega) n > m \Rightarrow g(n) = f(n)\},$$

then $\{E(f): f \in \Pi_{n\in\omega}X_n\}$ forms a partition of $\Pi_{n\in\omega}X_n$ and the resultant quotient set is denoted by $\nabla_{n\in\omega}X_n$. If $S \subseteq \Pi_{n\in\omega}X_n$, we let $E(S)$ denote its image in $\nabla_{n\in\omega}X_n$.

Lemma (Kunen [3 and 6, pg. 58]). *Suppose* X_n *is a compact Hausdorff space for each* $n \in \omega$ *and* $\nabla_{n\in\omega}X_n$ *has the quotient topology induced by* $\sqcup_{n\in\omega}X_n$, *then*

(i) G_δ*-sets in* $\nabla_{n\in\omega}X_n$ *are open*

(ii) $\sqcup_{n\in\omega}X_n$ *is paracompact iff* $\nabla_{n\in\omega}X_n$ *is paracompact*[2]

(iii) Every open cover of $\nabla_{n\in\omega}X_n$ *has a subcover of cardinality* $\leq c$ *(the cardinality of the continuum) whenever* X_n *is scattered* $\forall \, n \in \omega$.

For $A, B \in \mathbf{P}(\omega)$ define $A \leq B$ if $A - B$ is finite; $A \equiv B$ if $A \leq B$ and $B \leq A$. Observe that $\equiv$ is an equivalence relation on $\mathbf{P}(\omega)$. Suppose λ is an ordinal and $f \in {}^{\omega}\lambda$, for each $A \in \mathbf{P}(\omega)$, we define in $\nabla^{\omega}(\lambda + 1)$, $\langle A, f\rangle = E(\Pi_{n\in\omega}A_f(n))$, where

[2]With (i) $\nabla_{n\in\omega}X_n$ is paracompact iff every open cover has a pairwise disjoint clopen refinement.

$$A_f(n) = \begin{cases} [f(n) + 1, \lambda] & \text{if } n \in A \\ [0, f(n)] & \text{if } n \notin A. \end{cases}$$

$\{\langle A,f\rangle : A \in \mathbf{P}(\omega)\}$ forms a clopen partition of $\nabla^\omega(\lambda + 1)$ since $A \equiv B$ iff $\langle A,f\rangle \cap \langle B,f\rangle \neq \emptyset$.

Theorem 1. *Suppose λ is an ordinal with* $\mathrm{cf}(\lambda) \neq \omega$, *then for* $\square^\omega(\lambda + 1)$ *to be paracompact it is necessary and sufficient that* $\square^\omega(\alpha + 1)$ *be paracompact* $\forall\ \alpha < \lambda$.

Proof. Necessity is obvious so we prove sufficiency only.

Without loss of generality, we assume λ is the supremum of an increasing sequence $\{\kappa_\alpha : \alpha < \mathrm{cf}(\lambda)\}$. Let R be an open cover of $\nabla^\omega(\lambda + 1)$. For each $\tau < \omega_1$ and $d \in {}^\tau c$ we construct inductively $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ to satisfy:

(1) $V(d)$ and $W(d)$ are clopen subsets of $\nabla^\omega(\lambda + 1)$, $\exists\ U \in R \ni V(d) \subseteq U$, $V(d) \cup W(d) \subseteq W(d \restriction \sigma)\ \forall\ \sigma < \tau$, and if $\sigma < \tau$ is a limit ordinal, then $W(d \restriction \sigma) = \bigcap_{\rho<\sigma} W(d \restriction \rho)$.

(2) If $\sigma \leq \tau$ is an odd ordinal[3], then

$$\{V(e) : \mathrm{dom}(e) \leq \sigma\} \cup \{W(e) : \mathrm{dom}(e) = \sigma\}$$

is a pairwise-disjoint covering of $\nabla^\omega(\lambda + 1)$.

(3) $A(d)$ is an infinite subset of ω and if $\sigma \leq \tau$ is a non-limit ordinal, then $A(d \restriction \sigma) \leq A(d \restriction \rho)\ \forall\ \rho < \sigma$.

(4) If $E(x) \in W(d)$ and $\phi < A \leq A(d \restriction \sigma)\ \forall\ \sigma \leq \tau$, then $E(\{y : x(n) \leq y(n) \leq \lambda \text{ if } n \in A,\ y(n) = x(n) \text{ if } n \notin A\}) \subseteq W(d)$.

(5) $\theta(d) \in {}^\omega\lambda$ is a constant function with values in $\{\kappa_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ and if $\sigma \leq \tau$ is even, then

$$\theta(d \restriction \sigma)(0) > \theta(d \restriction \rho)(0)\ \forall\ \rho < \sigma.$$

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[3] σ is an odd ordinal when $\sigma = \sigma_0 + 2n + 1$, where $\sigma_0 = 0$ or is a limit ordinal and $n \in \omega$. If σ is not odd it is even.

(6) If $\sigma \leq \tau$ is odd, then $W(d \restriction \sigma) \leq \langle A(d \restriction \sigma), \theta(d \restriction \sigma)\rangle$,

(7) If $\sigma \leq \tau$ is a non-limit even ordinal and $\rho = \sigma - 1$, then $\exists$ a clopen subset $G(d \restriction \sigma)$ of $\nabla_{n \notin A(d \restriction \rho)}(\theta(d \restriction \sigma)(n) + 1)$ such that

$$V(d \restriction \sigma) = W(d \restriction \sigma) \cap \langle A(d \restriction \rho), \theta(d \restriction \sigma)\rangle \text{ and}$$

$$W(d \restriction \sigma) = \{E(x) \in W(d \mid \rho): E(x \restriction \omega - A(d \restriction \rho)) \in G(d \restriction \sigma)\}.$$

Now suppose our objects $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ have been constructed to satisfy (1) through (7) $\forall d \in {}^{\tau}c$ $\forall \tau < \omega_1$. If $E(x) \notin \cup\{V(t \restriction \tau): t \in {}^{\omega_1}c, \tau < \omega_1\}$ then by (1) and (2) we may find for each $\tau < \omega_1$, $d_\tau \in {}^{\tau}c$ such that $E(x) \in W(d_\tau)$. Again from (1) and (2), if $\sigma < \tau$ is odd and $d \in {}^{\sigma}c$ such that $d \neq d_\tau \restriction \sigma$, then $E(x) \notin W(d)$; therefore, $\sigma < \tau \Rightarrow d_\sigma = d_\tau \restriction \sigma$. From (5) we may find the first even ordinal $\rho < \omega_1$ such that for every n,

$$x(n) > \theta(d_\rho)(0) \Rightarrow x(n) \geq \sup_{\tau < \omega_1} \theta(d_\tau)(0).$$

From (6) $\exists\, y \in \square^{\omega}(\lambda + 1) \ni E(y) = E(x)$ and

$$A(d_{\rho+1}) = \{n: y(n) > \theta(d_{\rho+1})(n)\}.$$

From (7) $E(y) \in V(d_{\rho+2})$, a contradiction. Therefore,

$$\{V(t \restriction \tau): t \in {}^{\omega_1}c, \tau < \omega_1\}$$

is a cover of $\nabla^{\omega}(\lambda + 1)$ and we are done, so we should begin our construction.

Let $A(\phi) = \omega$, $W(\phi) = \nabla^{\omega}(\lambda + 1)$, and α be the first ordinal such that $E(\Pi^{\omega}[\kappa_\alpha, \lambda])$ is contained in some $U \in R$. Let $\theta(\phi)(n) = \kappa_\alpha$ $\forall n \in \omega$ and $V(\phi) = \langle A(\phi), \theta(\phi)\rangle$.

Suppose for an ordinal $\rho < \omega_1$ we have constructed $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ to satisfy (1) through (7) $\forall d \in {}^{\tau}c$ $\forall$ $\tau < \rho$. Our construction at ρ needs three cases:

Case 1. ρ is an odd ordinal

Let $\tau = \rho - 1$ and $\theta(e) = \theta(d)$ if $e \in {}^{\rho}c$ and $e \restriction \tau = d$. Let

$$\{A(e): e \in {}^{\rho}c,\ e \upharpoonright \tau = d\}$$

be a listing of exactly one element chosen from each equivalence class of elements of

$$\{A: \phi < A < A(d \upharpoonright \sigma),\ \sigma \leq \tau\}.$$

For each $e \in {}^{\rho}c$ we let

$$W(e) = W(e \upharpoonright \tau) \cap \langle A(e), \theta(e) \rangle .$$

If $d \in {}^{\tau}c$, then $W(d) \cap \langle \phi, \theta(d) \rangle$ is a clopen subset of $E(\Pi^{\omega}[0, \theta(d)(0)])$; therefore, by the lemma (ii) and (iii) we may find a pairwise-disjoint clopen refinement of R

$\{V(e): e \in {}^{\rho}c,\ e \upharpoonright \tau = d\}$ whose union is $W(d) \cap \langle \phi, \theta(d) \rangle$.

Clearly (1) through (7) are satisfied.

Case 2. ρ is a non-limit even ordinal.

Let $\tau = \rho - 1$ and $A(e) = A(d)$ if $e \in {}^{\rho}c$ and $e \upharpoonright \tau = d$. If $d \in {}^{\tau}c$ and $W(d) = \phi$, we let $W(e) = V(e) = \phi$ and

$$\theta(e)(n) = n_{\alpha} \text{ if } \theta(d)(n) = n_{\alpha-1} \quad \forall\, n \in \omega$$

If $d \in {}^{\tau}c$ and $W(d) \neq \emptyset$, let

$$Y^*(d) = \{g: g^{-1}(\lambda) = A(d),\ E(g) \in W(d)\}.$$

We will wish to cover $Y^*(d)$ by

$$\cup\{W(e): e \upharpoonright \tau = d\}.$$

From (4), $Y(d) = \{g \upharpoonright \omega - A(d): g \in Y^*(d)\} \neq \emptyset$.

In $\nabla_{n \notin A(d)}(\theta(d)(n) + 1)$, let

$R(d) = \{E(\Pi_{n \notin A(d)} U(n)): E(\Pi_{n \in \omega} U(n)) \subseteq \text{some } U \in R,$

$E(\Pi U(n)) \cap Y^*(d) \neq \emptyset\}.$

From (5) of the induction hypothesis and the lemma, (ii) and (iii), $\exists$ a pairwise disjoint clopen refinement $\{G(\gamma): \gamma < c\}$ of $R(d)$ whose union is $E(Y(d))$. If $e \in {}^{\rho}c$, $e \upharpoonright \tau = d$, $e(\tau) = \gamma$, then let

$$W(e) = \{E(x) \in W(d): E(x \upharpoonright \omega - A(d)) \in G(\gamma)\}.$$

For each γ we may find $n_{\alpha(\gamma)} > \theta(d)(0)$ such that $\{E(x)\in W(d): E(x \upharpoonright \omega - A(d)) \in G(\gamma)$ and $x(n) > n_{\alpha(\gamma)}$ $\forall$ but finitely many $n \in A(d)\} \subseteq$ some $U \in R$.
Let $\theta(e)(n) = n_{\alpha(\gamma)}$ $\forall$ $n\in\omega$ and $V(e) = W(e) \cap \langle A(d), \theta(e)\rangle$. Certainly (1) through (7) are satisfied.

Case 3. ρ is a limit ordinal.

If $e \in {}^{\rho}c$, let $A(e) = \omega$, $V(e) = \phi$, and find the first $\alpha < \omega_1 \ni n_\alpha > \theta(e \upharpoonright \tau)(0)$ $\forall$ $\tau < \rho$. We choose $\theta(e)(n) = n_\alpha$ $\forall$ $n\in\omega$. To satisfy (1) through (7) we observe that (i) of the lemma allows

$$W(e) = \cap_{\tau<\rho} W(e \upharpoonright \tau)$$

to be clopen.

The proof to Theorem 1 is completed.

If ${}^{\omega}\omega$ is ordered by $f < g$ if $\{n: g(n) \leq f(n)\} \equiv \phi$, then for an ordinal k, a k-scale is an order-preserving injection $s: k \to {}^{\omega}\omega$ such that $\{s(\alpha): \alpha < k\}$ is cofinal in ${}^{\omega}\omega$. Recall [2,7] that CH $\Rightarrow$ $\exists$ an ω_1-scale; MA $\Rightarrow$ $\exists$ a c-scale; an ω-scale; $\exists$ a k-scale and ℓ-scale $\Rightarrow$ $cf(k) = cf(\ell)$; for every model m with regular ordinals k and ℓ with $cf(k) \neq \omega \neq cf(\ell)$ and $k \leq \ell$, there is a model $n \supseteq m$ with a k-scale in ${}^{\omega}\omega$ and $c = \ell$; and $\exists$ models m of ZFC without k-scales for any k.

Theorem 2. *(* $\exists$ *a k-scale in* ${}^{\omega}\omega$*). Suppose* $cf(\lambda) = \omega$, *then for* $\sqcup^{\omega}(\lambda + 1)$ *to be paracompact it is necessary and sufficient that* $\exists \{\gamma_n: n\in\omega\} \subseteq \lambda \ni \sup_{n\in\omega}\gamma_n = \lambda$ *and* $\dot{\sqcup}_{n\in\omega}(\gamma_n + 1)$ *is paracompact.*

Proof. Necessity is obvious so we prove sufficiency.

WLOG assume $\gamma_n < \gamma_{n+1}$ $\forall$ $n\in\omega$, $cf(\gamma_n) = 1$ $\forall$ $n\in\omega$, and $\{s(\alpha): \alpha < k\}$ is a k-scale in ${}^{\omega}\omega$ for a regular k. Let R be an open cover of $\nabla^{\omega}(\lambda + 1)$. For each $\tau < k$ and $d \in {}^{\tau}c$ we construct inductively $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ to satisfy:

(1) $V(d)$ and $W(d)$ are clopen subsets of $\nabla^{\omega}(\lambda + 1)$, $\exists\, U \in R \ni$ $V(d) \subseteq U$, $V(d) \cup W(d) \subseteq W(d \restriction \sigma)$ $\forall\, \sigma < \tau$, and if $\sigma < \tau$ is a limit ordinal $W(d \restriction \sigma) = \bigcap_{\rho<\sigma} W(d \restriction \rho)$.

(2) If $\sigma \leq \tau$ is an odd ordinal, then

$$\{V(e)\colon \mathrm{dom}(e) \leq \sigma\} \cup \{W(e)\colon \mathrm{dom}(e) = \sigma\}$$

is a pairwise-disjoint covering of $\nabla^{\omega}(\lambda + 1)$.

(3) $A(d)$ is an infinite subset of ω and if $\sigma \leq \tau$ is a non-limit ordinal, then $A(d \restriction \sigma) \leq A(d \restriction \rho)$ $\forall\, \rho < \sigma$.

(4) $\theta(d)(n) = \gamma_{s(\alpha)(n)}$ $\forall\, n \in \omega$ and some $\alpha < k$; and if $\sigma \leq \tau$ is even, then

$$\{n\colon \theta(d \restriction \sigma)(n) \leq \theta(d \restriction \rho)(n)\} \equiv \phi \qquad \forall\, \rho < \sigma.$$

(5) If $\sigma \leq \tau$ is odd, then $W(d \restriction \sigma) \subseteq \langle A(d \restriction \sigma), \theta(d \restriction \sigma)\rangle$ and $\{V(e)\colon e \in {}^{\sigma}c, e \restriction \sigma - 1 =\!\!\rceil\, d \restriction \sigma - 1\} = \langle \phi, \theta(d \restriction \sigma)\rangle \cap$ $W(d \restriction \sigma - 1)$.

(6) If $\sigma \leq \tau$ is a non-limit even ordinal, then

$$V(d \restriction \sigma) = W(d \restriction \sigma) \cap \langle A(d \restriction \sigma - 1), \theta(d \restriction \sigma - 1)\rangle.$$

Now suppose our objects $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ have been constructed to satisfy (1) through (6) $\forall\, d \in {}^{\tau}c$ $\forall\, \tau < k$. For $x \in \Pi^{\omega}(\lambda + 1)$ define

$$x^{\#}(n) = \begin{cases} 0 \text{ if } x(n) = \lambda \\ x(n) \text{ otherwise.} \end{cases}$$

We may find the first $\alpha \ni \{n\colon \gamma_{s(\alpha)(n)} \leq x^{\#}(n)\} = \phi$. If $\alpha = \alpha_0 + m$, where $\alpha_0 = 0$ or is a limit ordinal and $m \in \omega$, let $\tau = \alpha_0 + 2(m + 1)$. From (2), (4), (5), and (6) we have

$$E(x) \in \cup\{V(e)\colon \mathrm{dom}(e) \leq \tau\}.$$

Therefore, $\{V(d)\colon d \in {}^{\tau}c, \tau < k\}$ is a pairwise-disjoint clopen refinement of R covering $\nabla^{\omega}(\lambda + 1)$. So we must complete our construction.

Let $A(\phi) = \omega$, $W(\phi) = \nabla^{\omega}(\lambda + 1)$, and α be the first ordinal such that $E(\Pi_{n\in\omega}[\gamma_{s(\alpha)(n)}, \lambda])$ is contained in some $U \in R$. Let $\theta(\phi)(n) = \gamma_{s(\alpha)(n)}$ $\forall$ $n\in\omega$ and $V(\phi) = \langle A(\phi), \theta(\phi) \rangle$.

Suppose for an ordinal $\rho < k$ we have constructed $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ to satisfy (1) through (6) $\forall$ $d\in {}^{\tau}c$ $\forall$ $\tau < \rho$. Our construction at ρ needs three cases:

Case 1. ρ is an odd ordinal.

Let $\tau = \rho - 1$ and $\theta(e) = \theta(d)$ if $e\in {}^{\rho}c$ and $e \restriction \tau = d$. Let

$$\{A(e): e\in {}^{\rho}c, e \restriction \tau = d\}$$

be a listing of exactly one element from each equivalence class of elements of

$$\{A: \phi < A < A(d \restriction \sigma), \sigma \leq \tau\}.$$

For each $e\in {}^{\rho}c$ we let

$$W(e) = W(e \restriction \tau) \cap \langle A(e), \theta(e) \rangle.$$

If $d\in {}^{\tau}c$, then $W(d) \cap \langle \phi, \theta(d)\rangle$ is a clopen subset of

$$E(\Pi_{n\in\omega}[0, \theta(d)(n)])$$

and $\Pi_{n\in\omega}[0, \theta(d)(n)]$ is a clopen subset of a subproduct of $\Pi_{n\in\omega}(\gamma_n + 1)$; therefore, by the lemma (ii) and (iii) we may find a pairwise-disjoint clopen refinement of R, $\{V(e): e\in {}^{\rho}c, e \restriction \tau = d\}$ whose union is $W(d) \cap \langle \phi, \theta(d) \rangle$. Clearly, (1) through (6) are satisfied.

Case 2. ρ is a non-limit even ordinal.

Let $\tau = \rho - 1$, and $A(e) = A(d)$, and $W(e) = W(d)$ if $e\in {}^{\rho}c$ and $e \restriction \tau = d$. If $d\in {}^{\tau}c$ and $W(d) = \phi$, we let $W(e) = V(e) = \phi$ and

$$\theta(e)(n) = \gamma_{s(\alpha+1)(n)} \ \forall\ n\in\omega;\ \text{where}$$

$$\theta(e \restriction \tau)(n) = \gamma_{s(\alpha)(n)} \ \forall\ n\in\omega.$$

If $d \in {}^{\tau}c$, $W(d) \neq \phi$, and

$$Y(d) = \{g \restriction \omega - A(d) \colon y^{-1}(\lambda) = A(d),\ E(y) \in W(d)\} = \phi.$$

In $\nabla_{n \notin A(d)}(\theta(d)(n) + 1)$, let

$$R(d) = \{E(\Pi_{n \notin A(d)} U(n)) \colon E(\Pi_{n \in \omega} U(n)) \subseteq \text{some } U \in R,$$
$$\exists\ E(g) \in W(d) \cap E(\Pi_{n \in \omega} U(n)), g^{-1}(\lambda) = A(d)\}$$

Since $\sqcup_{n \notin A(d)}(\theta(d)(n) + 1)$ is homeomorphic to a clopen subset of a subproduct of $\sqcup_{n \in \omega}(\gamma_n + 1)$, we may use the lemma, (ii) and (iii), to find a pairwise disjoint clopen refinement $\{G(\delta) \colon \delta < c\}$ of $R(d)$ whose union is $E(Y(d))$. If $e \in {}^{\rho}c$, $e \restriction \tau = d$, $e(\tau) = \delta$, then let $\alpha(\delta)$ be the first ordinal $> \alpha(d)$, where $\theta(d)(n) = \gamma_{s(\alpha(d))(n)}\ \forall\ n \in \omega$, such that

$$V(e) = \{E(x) \in W(d) \colon E(x \restriction \omega - A(d)) \in G(\delta),\ x(n) >$$
$$\gamma_{s(\alpha(\delta))(n)}\ \forall\ n \in \omega\}$$

is contained in a member of R. Let $\theta(e)(n) = \gamma_{s(\alpha(\delta)(n)}\ \forall\ n \in \omega$. Clearly, (1) through (6) are satisfied.

Case 3. ρ is a limit ordinal.

If $e \in {}^{\rho}c$, let $A(e) = \omega$, $V(e) = \phi$, and $\theta(e)(n) = \gamma_{s(\alpha)(n)}$ $\forall\ n \in \omega$, where

$$\alpha = \sup\{\beta \colon \theta(e \restriction \tau)(n) = \gamma_{s(\beta)(n)}\ \forall\ n \in \omega,\ \tau < \rho\}.$$

To see that (1) through (6) are satisfied, we must show

$$W(e) = \cap_{\tau < \rho} W(e \restriction \tau) \text{ is open.}$$

However, if $E(x) \in W(e)$, then the induction hypothesis and the definition of $W(d)$ in Case 2 yields

$$E(\Pi[x^*(n), x(n)]) \subseteq W(e),$$

where

$$x^*(n) = \begin{cases} x(n) \text{ if } cf(x(n)) = 1 \\ \theta(e)(n+1) \text{ if } x(n) \text{ is a limit } > \theta(e)(n) \\ \sup\{\theta(e \restriction \tau)(n) \colon \theta(e \restriction \tau)(n) < x(n),\ \tau < \rho\} + 1, \text{ other-} \\ \text{wise.} \end{cases}$$

This completes the construction and the proof of Theorem 2.

Remarks

A. There are many models of ZFC, constructed via forcing, in which there are no $\mathfrak{k}$-scales [2]. However, J. Roitman [4] has shown that in some of these models, techniques inadvertedly, in some sense, yield $\square_{n\in\omega} X_n$ paracompact $\forall$ compact metrizable X_n; specifically she has shown:

In a model m of set theory which is a direct iterated CCC extension of length $\mathfrak{k}$ of a model n, $\mathrm{cf}(\mathfrak{k}) > \omega \Rightarrow \nabla_{n\in\omega} X_n$ is paracompact if X_n is regular and separable.

A simple adaptation of her proofs will give the conclusion of Theorem 2 in m.

B. Suppose u_0 is an ordinal and for $n > 0$ u_n is the lexicographic ordered product of u_{n-1} with itself. Let $u = \sup_{n\in\omega} u_n$. It is unknown whether ($\exists$ a c-scale in ${}^{\omega}\omega$) $\Rightarrow \square^{\omega}(u + 1)$ is paracompact when $u_0 = \omega_1$; however, our theorems show ($\exists$ a $\mathfrak{k}$-scale in ${}^{\omega}\omega$) $\Rightarrow \square^{\omega}(u_n + 1)$ is paracompact $\forall\ n \in \omega$. It is unknown whether $\square^{\omega}(\omega + 1)$ is paracompact $\Rightarrow \square^{\omega}(u + 1)$ is paracompact when $u_0 = \omega$; although $\square^{\omega}(u_n + 1)$ is paracompact for each n.[4] The simplest question still unanswered is "Does there exist a model m of ZFC in which $\square^{\omega}(\lambda + 1)$ is not paracompact for some ordinal λ?" The hardest question asks that $\lambda = \omega$.

C. We observe a recent result communicated to the author by E. K. van Douwen: If X_n is compact $\forall\ n\in\omega$, then $\square_{n\in\omega} X_n$ is pseudo-normal. The author gives much appreciation to the referee whose suggestions for clarification of

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[4] $\nabla^{\omega}(u_n + 1)$ may be embedded in $\nabla^{u_n + 1}(\omega + 1)$.

unnecessary technicalities in our proofs appear.

Added in proof

Recently, J. Roitman has proved that $\square_{n\in\omega}X_n$ is paracompact whenever each X_n is compact first countable and ${}^{\omega}\omega$ fails to have a cofinal family of cardinality less than the continuum. A corollary to this theorem and our theorems 1 and 2 yields $c = \omega_2 \Rightarrow \square^{\omega}\omega_1 + 1$ is paracompact. Independently, I have shown the same corollary and, in addition:

Suppose, in theorem 2, ($\exists$ a κ-scale in ${}^{\omega}\omega$) is replaced by κ is the least cardinal of any cofinal family in ${}^{\omega}\omega$ and $\mathcal{A} \subset \mathbf{P}(\omega)$ with $|\mathcal{A}| = \kappa$, then

$$E(\{x \in \square^{\omega}\lambda + 1: x^{-1}(\lambda) \in \mathcal{A}\}) \text{ is paracompact.}$$

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