TOPOLOGY PROCEEDINGS

Volume 7, 1982

Pages 301–327

http://topology.auburn.edu/tp/

ORDERABLE SUBSPACES OF ČECH-REMAINDERS

by

Scott Williams

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog.edu
ISSN:	0146-4124

ORDERABLE SUBSPACES OF ČECH-REMAINDERS[1]

Scott Williams[1]

In 1963 I. Parovičenko obtained the following: *Assume the continuum hypothesis. The space* $(\Pi^{¢}2)_{¢}$ *is homeomorphic to a dense subspace of* $\beta\omega-\omega$ [Pa]. Subsequent improvements and extensions of this theorem were presented in [CN1], [HN], and [Sm]. In each case we find (1) a set-theoretic hypothesis known to be consistent with and independent of ZFC is assumed, and (2) the dense subspace of the Čech-Stone remainder is *orderable* (i.e., homeomorphic to a linearly ordered topological space). There is, however, a related theorem in ZFC: If X is a realcompact, locally compact, non-compact space of π-weight at most ¢, then $\beta X-X$ possesses a dense subspace the pre-image of an orderable space under a perfect irreducible map [Wi2]. In this paper we now present

1.5. *The following are mutually consistent with the axioms of* ZFC: $\exists$ *a* P-*point;* $\exists$ *an* ω_1-*scale; if* X *is a locally compact, non-pseudocompact space, then* $\beta X-X$ *has no dense orderable subspace.* (Thus establishing the [Wi2] result as "best possible" in ZFC.)

2.5. *Assume* $\exists$ *a strong* $P\kappa$-*point and* $\exists$ *a* κ-*scale. If* X *is a paracompact, locally compact, non-compact first*

[1]In part, this paper was conceived while its author was an NRC-Ford Foundation Senior Postdoctoral Fellow (1980-81). The results were presented at the Annual A.M.S. Winter Meetings (San Francisco; January 1981) and at the Annual Spring Topology Conference (Blacksburg, Virginia, March 1981).

countable space, then $\beta X-X$ *contains a dense orderable subspace.* (Compared to the [Sm] theorem, our assumptions are much weaker and their consequences much stronger.)

3.1. *The continuum hypothesis is equivalent to the statement: If* X *is a* σ*-compact, locally compact, non-compact separable space, then* $\beta X-X$ *has a dense orderable subspace.* (This contains the converse to the [CN1] theorem.)

3.2. *The following are mutually consistent with the axioms of* ZFC*:* $\omega_1<\mathfrak{c}$*;* $\exists$ *an* ω_1*-scale; if* X *is a locally compact, non-compact metrizable space of weight at most* $\mathfrak{c}$*, then* $(\Pi^{\omega_1}2)_{\omega_1}$ *embeds densely into* $\beta X-X$. (This approximates the [CN1] theorem without CH).

0. Conventions

0.1. We assume ZFC. All ordinals and cardinals have the von Neuman definition so $2 = \{0,1\}$ and $0 < 1 < 2$. If α is an ordinal, ω_α is the α'th ordinal (while $\omega = \omega_0$). $f^{\leftarrow}$ denotes the inverse relation of a function f. For sets X and Y, ${}^XY$ is the set of functions from X to Y, but if X is a cardinal κ we write $exp(\kappa)$ for ${}^\kappa 2$. When X is a set $|X|$ is the cardinality of X. If X is a set and κ is a cardinal, then

$$[X]^\kappa = \{Y \subseteq X: |Y| = \kappa\}.$$

($[X]^{<\kappa}$ and $[X]^{\leq\kappa}$ are defined similarly.) So $\mathfrak{c} = |exp(\omega)| = |[\omega]^\omega|$.

0.2. *Additional axioms:* CH is the hypothesis $\mathfrak{c} = \omega_1$ and MA is Martin's axiom. $\exists$ *a* P*-point* is the statement:

there is a free ultrafilter u on ω satisfying

(1) $\{U_n: n\in\omega\} \subseteq u \Rightarrow \exists\ U\in u,\ |U-U_n| < \omega \quad \forall\ n\in\omega$.

For a regular uncountable cardinal $\kappa\leq\mathfrak{c}$, *$\exists$ a strong Pκ-point* is the statement: there is a function A: $\kappa \to [\omega]^\omega$ satisfying

(2) $A(\kappa)$ generates a free ultrafilter, and

(3) $\mu<\nu<\kappa \Rightarrow |A(\nu) - A(\mu)| < \omega$.

A *strong* P-*point* is a strong $P\omega_1$-point (the ultrafilter generated by A). For an uncountable cardinal $\kappa\leq\mathfrak{c}$, *$\exists$ a κ-scale* is the statement: there is a function s: $\kappa \to {}^\omega\omega$ satisfying

(4) $r \in {}^\omega\omega \Rightarrow \exists\ \nu<\kappa,\ |\{n\in\omega: s(\nu)(n) \leq r(n)\}| < \omega$, and

(5) $\mu<\nu<\kappa \Rightarrow |\{n\in\omega: s(\nu)(n) \leq s(\mu)(n)\}| < \omega$.

(See [He] and [Ru] for more on scales.)

0.3. In most cases it is superfluous to consider a space to be less than T_1 and normal, so we assume all spaces have these properties. If X is a space, int_X and cl_X (or just *int* and *cl*) denote the interior and closure operations. We denote (with no confusion about zero sets) $Z(X)$ for the lattice of closed subsets of X, and βX for the space of ultrafilters in $Z(X)$. For a subset A of X we set $A^* = (cl_{\beta X}(A)) - X$. We abbreviate *nbhd* for (not necessarily open) neighborhood, and for $x \in X$, the *point-character* is $\chi(x,X)$ equal the least cardinality of a base for a nbhd system at x. A π-*base* for a space (X,τ) is any cofinal subset of the poset $(\tau-\{\phi\},\supseteq)$. For a collection $\{X(\nu): \nu<\kappa\}$ of spaces we use $\Pi\{X(\nu): \nu>\kappa\}$, or $\Pi\langle X(\nu)\rangle$, for the Tychonoff product of the spaces; the projection maps are π_ν. Objects like $\Pi exp(\kappa)$ should offer no confusion. When

(X,τ) is a space and κ is a cardinal, $(X)_\kappa$ is the space generated from the union of τ with the set of all intersections of less than κ many open sets.

0.4. *Order:* Suppose X is a space made orderable by a linear ordering $\leq$; further, suppose $\kappa = \omega_\alpha$ is a regular cardinal. Then X is called (traditionally from [Si]) an η_α-*set* if for every $A\times B \subseteq X\times X$ such that $inf\{|A|,|B|\} < \kappa$ and $a<b\ \forall\ (a,b) \in A\times B$, we may find an $x \in X$ with $a<x<b\ \forall\ (a,b) \in A\times B$. An η_α-set X is called a *true* η_α-*set* if $\chi(x,X) = \omega_\alpha$ $\forall x \in X$. When $\kappa = \omega_\alpha$ is regular, an example of a true η_α-set is $(\Pi exp(\kappa))_\kappa$. The *canonical* η_α-*set* is the space

$(\{f\in\Pi exp(\kappa)\colon f \text{ is constant on a final segment of } \kappa\})_\kappa$.

1. No Dense Orderable Subspaces

A standard result in topology shows that the only dyadic spaces possessing a dense orderable subspace are the continuous image of the Cantor set. From that result we isolate a property having varied applications.

1.1. A family $\mathcal{I}$ of subsets of a topological space will be called *independent* whenever

(1) $\mathcal{J},\mathcal{K} \in [\mathcal{I}]^{<\omega}$ and $int(\cap\mathcal{J}) \subseteq cl(\cup\mathcal{K})$

implies $\mathcal{J} \cap \mathcal{K} \neq \emptyset$.

An independent family $\mathcal{I}$ will be called *bounded* whenever

(2) $|\mathcal{I}| > sup\{|\mathcal{J}|^+\colon \mathcal{J} \subseteq \mathcal{I}$ and either $int(\cap\mathcal{J}) \neq \emptyset$ or $cl(\cup\mathcal{J}) \neq X\}$.

1.2. *Lemma: Suppose that a space* X *has a bounded independent family of clopen sets. Then every orderable* *subspace of* X *is nowhere dense.*

Proof. By way of contradition, we suppose D is a dense orderable subspace of a non-empty open set G of X and that $\mathcal{I}$ is an unbounded independent family of clopen sets of X. For a given $x \in D$ we use DeMorgan's laws to determine another such family

$$\mathcal{I}(x) = \{I \in \mathcal{I}: x \in I\} \cup \{X-I: x \notin I \in \mathcal{I}\}.$$

Since D is a linearly ordered topological space, we may also find regular cardinals λ and ρ, and order preserving injections $f: \lambda \to D$ and $g: \rho \to D$ whose images are cofinal, respectively, in

$$\{y \in D: y < x\} \text{ and } \{y \in D: x < y\}.$$

We therefore have a map $h: \mathcal{I}(x) \to \lambda\times\rho$ satisfying

(1) $\{d \in D: (f\circ\pi_\lambda\circ h)(I) < d < (g\circ\pi_\rho\circ h)(I)\} \subseteq I$ for each $I \in \mathcal{I}(x)$.

If κ is a regular cardinal, $|\lambda\times\rho| < \kappa \leq |\mathcal{I}(x)|$, then we may find $(\alpha,\beta) \in \lambda\times\rho$ so that $|h^{\leftarrow}(\alpha,\beta)| = \kappa$. From (1) we find $int(\cap h^{\leftarrow}(\alpha,\beta)) \neq \emptyset$ (since D is dense in G). So, by 1.1(2), $|\mathcal{I}(x)| \leq |\lambda\times\rho|$.

If $\mathcal{J} \subseteq \mathcal{I}(x)$ and $|\mathcal{J}| < \lambda$, then left-side of the equation in (1) shows that $int(\cap\mathcal{J}) \neq \emptyset$. Using the right-side, the same conclusion is made when $|\mathcal{J}| < \rho$. Since $\mathcal{I}(x)$ is unbounded, $|\mathcal{I}(x)| > |\lambda\times\rho|$.

1.3. *The algebra for the simultaneous addition of random reals* (see [So] or [TZ, pg. 173]): Let $\mathcal{M}$ be a c.t.m. ZFC (for countable transitive model of ZFC). For a set $w \in \mathcal{M}$, $\mathbf{B}(w)$ designates the σ-complete algebra of all Borel sets of $\Pi(exp(w\times\omega))$. Generate a [0,1]-measure m on

$\mathbf{B}(w)$ with the base

$$m(\pi^{\leftarrow}_{(\nu,n)}(i)) = \frac{1}{2} \qquad \forall(\nu,n,i) \in w\times\omega\times 2.$$

For $B_0, B_1 \in \mathbf{B}(w)$ define the equivalence

$$\langle B_0 \rangle = \langle B_1 \rangle \text{ if } m((B_1 - B_0) \cup (B_0 - B_1)) = 0.$$

The desired algebra is the resulting quotient algebra, we designate by $Q(w)$. $Q(w)$ is a measure algebra (and hence, it is a c.c.c. complete Boolean algebra), so forcing with it preserves cardinals. If $w = \emptyset$, $Q(w)$ is the two element algebra.

In the sequel $\mathfrak{M}$ is always a c.t.m. of ZFC. The lemma is straightforward.

1.4. *Lemma: Suppose* $w, \kappa \in \mathfrak{M}$ *and* $\phi \neq w \subsetneq \kappa$. *Recursively, on the Borel listing of* $\mathbf{B}(w)$, *define* $\phi\colon Q(w) \to Q(\kappa)$ *beginning with*

$$\phi(\langle \pi^{\leftarrow}_{(\eta,n)}(i) \rangle_{Q(w)}) = \langle \pi^{\leftarrow}_{(\eta,n)}(i) \rangle_{Q(\kappa)} \qquad \forall(\eta,n,i) \in w\times\omega\times 2.$$

Then (1). ϕ *is a measure preserving isomorphism of* $Q(w)$ *onto its image in* $Q(\kappa)$.

(2). *If* D *is a dense set in* $Q(w)$, *then* D *is a dense set in* $Q(\kappa)$.

1.5. *Theorem: Suppose* $\mathfrak{M} \models$ CH. *If* $\kappa \in \mathfrak{M}$ *is a regular cardinal,* $\kappa > \omega_1$, *and if* G *is a* $Q(\kappa)$*-generic ultrafilter over* $\mathfrak{M}$, *then* $\mathfrak{M}[G] \models$

(1). $\exists$ *a* P*-point.*

(2). $\exists$ *an* ω_1*-scale.*

(3). *If* X *is a locally compact non-pseudocompact space, then* X^* *has no dense orderable subspace.*

Proof. *ad*(1) is due to K. Kunen and P. E. Cohen (unpublished). *ad*(2) due to Solovay (see footnote in [He]), follows from [TZ, pg. 176-177].

ad(3). Suppose (in any model) X is a locally compact non-pseudocompact space, then there is an open set Y of X* the pre-image of ω^* under an open continuous surjection [vD1]. Clearly, Y has a bounded independent family of clopen sets whenever ω^* has one. So we will show ω^* has a bounded independent family of clopen sets, and then apply 1.2 to Y.

For all $(\nu,i) \in \kappa\times 2$ we set

$I(\nu,i) = \{n\in\omega: \langle \pi^{\leftarrow}_{(\nu,n)}(i)\rangle \in G\}$.

For each $f \in \mathcal{M} \cap {}^{\omega}(\kappa\times\omega)$ and $r \in \mathcal{M} \cap exp(\omega)$,

(4). $\{\langle \pi^{\leftarrow}_{f(n)}(r(n))\rangle: n \in \omega\}$ is dense in $Q(\kappa)$ since $m(\cup\{\pi^{\leftarrow}_{f(n)}(r(n)): n \in \omega\}) = 1$.

Now suppose $\emptyset \neq F \in [\kappa]^{<\omega}$ and $s \in exp(F)$ are such that

(5). $A = \cap\{I(\nu,s(\nu)): \nu \in F\} \in \mathcal{M}$.

Then for every $n\in\omega-A$ we may find a $\nu(n) \in F$ satisfying

$[n \notin I(\nu(n),s(\nu(n)))] \in G$.

If $\omega-A$ is infinite, we may find a $\lambda \in F$ so that

$N = \{n \in \omega: \nu(n) = \lambda\} \in \mathcal{M} \cap [\omega]^{\omega}$.

Therefore, $N \cap I(\lambda,s(\lambda)) = \emptyset$. If $\omega-A$ is finite, then $I(\nu,1-s(\nu))$ is finite $\forall\ \nu \in F$. However, for any $\nu \in \kappa$ and any finite $\{a(n): n \in \omega\} \in \mathcal{M} \cap [\omega]^{\omega}$, we set, in (4) $f(n) = (\nu,a(n))\ \forall\ n \in \omega$ to see that $\forall\ i \in 2$,

(6). $\{a(n) \in I(\nu,i): n \in \omega\} \in \mathcal{M}[G] \cap [\omega]^{\omega}$.

So (5) is false, and, in particular $\mathcal{M}[G] \models$

(7). $\mathcal{I} = \{I(\nu,0)^*: \nu < \kappa\}$ is an independent family of clopen sets of ω^*.

Now suppose $v \in \mathcal{M}[G] \cap [\kappa]^{\omega_1}$ and $A \in \mathcal{M}[G] \cap [\omega]^{\omega}$.

From 1.4 there is a $w \in \mathcal{M} \cap [\kappa]^\omega$ such that

$$[n \in A] \in G_w = G \cap Q(w) \;\forall\; n \in A.$$

So $A \in \mathcal{M}[G_w]$. As cardinals are preserved, we may find a $\lambda \in \mathcal{M} \cap (v-w)$ such that $I(\lambda,i) \notin \mathcal{M}[G_w] \;\forall\; i \in 2$ (note that if $I(\lambda,i) \in \mathcal{M}[G_w]$, then

$$G_w \cup \{\langle \cup\{\overleftarrow{\pi}_{(\lambda,n)}(i): n \in I(\lambda,i)\}\rangle\}$$

extends G_w in Q, and so it is G_w. Thus, $\lambda \in w$.) Now apply (6) with $\mathcal{M}$ replaced by $\mathcal{M}[G_w]$ and κ replaced by $w \cup \{\lambda\}$ in order to see that

$$A \cap I(\lambda,i) \in [G \cap Q(w \cup \{\lambda\})] \cap [\omega]^\omega \;\forall\; i \in 2.$$

Therefore, $[G] \models$

$$A^* \not\subseteq \cap\{I(\nu,i)^*: \nu \in v\} \;\forall\; i \in 2.$$

Clearly, $\mathcal{I}$ is bounded.

We complete this section with a class of σ-compact, locally compact, non-compact, zero-dimensional spaces whose remainders never possess a dense orderable subspace.

1.6. *Example: For an uncountable cardinal* κ *consider the space*

$$\mathcal{D}(\kappa) = (\Pi exp(\kappa)) \times \omega.$$

If $\mathcal{D}(\kappa)^*$ *has a dense orderable subspace, then* $\kappa = \omega_1$ *and* $\exists$ *a strong* P*-point.*

Proof. Set $\mathcal{K} = \{(\overleftarrow{\pi}(i)\times\omega)^*: (\nu,i) \in \kappa \times 2\}$. Then the intersection of any uncountable subset of $\mathcal{K}$ has empty interior in $\mathcal{D}(\kappa)^*$ [vDvM] (their proof is for $\kappa = \mathfrak{c}$, but it extends for our needs). Since $\{\overleftarrow{\pi}_\nu(0): \nu < \kappa\}$ is an (bounded) independent family of clopen subsets of $\Pi exp(\kappa)$,

$$\mathcal{I} = \{(\overleftarrow{\pi}_\nu(0)\times\omega)^*: \nu < \kappa\}$$

is an independent family of clopen subsets of $\mathcal{D}(\kappa)^*$. From 1.2, $\mathcal{I}$ is not a bounded independent family. Therefore, $\kappa = \omega_1$.

Now suppose x belongs to a dense orderable subspace D of $\mathcal{D}(\omega_1)^*$ and, for symmetry, $x \in \cap\mathcal{I}$. Since $int(\cap\mathcal{I}) = \emptyset$, it is clear that $\chi(x,D) = \omega_1$. Just like $\mathcal{D}(\omega_1)^*$, D can have no convergent sequences. Therefore, in D (consider the λ and κ of 1.2), and, hence, in $\mathcal{D}(\omega_1)^*$, x has a well-ordered, by $\supseteq$, nbhd base $\mathcal{O} = \{0^*_\alpha: \alpha < \omega_1\}$. Via zero-dimensionality, we may assume $0_\alpha \cap (\Pi exp(\omega_1) \times \{n\})$ is clopen in $\Pi exp(\omega_1) \times \{n\}$ $\forall$ $(\alpha,n) \in \omega_1 \times \omega$. It is clear that

$$\{\{n \in \omega: 0_\alpha \cap (\Pi exp(\omega_1) \times \{n\}) \neq \emptyset\}^*: \alpha < \omega_1\}$$

is a linearly ordered, by $\supseteq$, nbhd base for

$$\{A \in [\omega]^\omega: (\Pi exp(\omega_1) \times A) \in x\} \in \omega^*.$$

1.7. *Remarks:*

(1). The reader should be warned that for w and κ, as in 1.4,

$$\mathcal{M}[G] \neq \mathcal{M}[Q(w) \cap G][Q(\kappa - w) \cap G].$$

In order to reach $\mathcal{M}[G]$ from $\mathcal{M}[Q(w) \cap G]$, one must first mod out the new point of Q(w) in $\mathcal{M}[Q(w) \cap G]$; namely, take $Q(\kappa)/N$, where N is the principle (in $\mathcal{M}[Q(w) \cap G]$) ideal complimentary to $Q(w) \cap G$.

(2). If, in 1.5, $Q(\kappa)$ is replaced by the Boolean algebra of clopen subsets of $\Pi exp(\kappa\times\omega)$, known as the κ-Cohen poset [Bu], then 1.5(1) is true (see [Ro2]), 1.5(2) is false (see [Ro1] or [TZ, pg. 177]), and 1.5(3) is true using a similar but easier argument (the $\neq$ in (1) above is =).

(3). Applying the long sought after P-point Independence Theorem [Wm], an easier proof of 1.5(3) can be found. Just modify the proof of 1.6 to show that: *If there is a locally compact non-pseudocompact space* X *such that* X^* *has a dense orderable subspace, then* $\exists$ *a* P-*point* (in ω^*).

2. Some Dense Orderable Subspaces

The purpose of this section is to exhibit dense orderable subspaces from relatively weak set-theoretic hypotheses. The first lemma makes it all easy.

2.1. *Lemma* [Wi_2], 1.12 and 2.3]*: Suppose* Y *is a space in which each non-empty* G_δ*-set has non-empty interior, and* Y *has* π*-weight at most* $\mathfrak{c}$*. If every element of* Y *has a well-ordered, by* $\supseteq$*,* nbhd *base, then* Y *has a dense orderable subspace.*

In order to guarantee that the subspaces we produce are dense, we apply a construction, used to study normality in box products implicitly due to Rudin, explicitly due to Kunen (see [Ru] and [vD2]).

2.2. *Definition:* Suppose $\{X(n): n \in \omega\}$ is a collection of spaces. For $f,g \in \Pi\{X(n): n \in \omega\}$ define the equivalence relation $E(f) = E(g)$ if

$|\{n \in \omega: f(n) \neq g(n)\}| < \omega.$

The *nabla product*, denoted by $\nabla\{X(n): n \in \omega\}$ or $\nabla\langle X(n)\rangle$ (or $\nabla^\omega Y$ if $X(n) = Y \ \forall\ n \in \omega$), is the ensuing quotient set topologized by the base of all sets of the form

$\nabla\langle G(n)\rangle = \{E(f): f \in \Pi\{G(n): n \in \omega\}\},$

where $G(n)$ is an open subset of $X(n)$ $\forall$ $n \in \omega$. Obviously, $\nabla G(n)$ is the open continuous image of the box product of the $X(n)$.

2.3. *Lemma: Suppose* $X = \cup\{K(n): n \in \omega\}$ *is a space in which* $K(n)$ *is compact and* $K(n) \subsetneq int(K(n+1))$ $\forall$ $n \in \omega$. *Then there is an open continuous map*

$$\Psi: \omega^* \times \nabla\langle K(n+1) - K(n)\rangle \to X^*$$

with dense image.

Proof. Set $X(n) = K(n+1) - K(n)$. For $u \in \omega^*$ and $f \in \Pi\langle X(n)\rangle$ let

$$\Psi(u,f) = \{Z \in \mathcal{Z}(X): \{n \in \omega: f(n) \in Z\} \in u\}.$$

Since each $X(n)$ has compact closure in X,

$$(u,E(f)) = (u,E(g)) \Rightarrow \Psi(u,f) = \Psi(u,g) \ \forall\ u \in \omega^*.$$

On the other hand, Ψ partitions its image since $\Psi(u,f) = \Psi(v,g)$ implies $u = v$.

To see that Ψ is an open function, suppose $Y \in \mathcal{Z}(X) - \Psi(u,f)$. Then $A \overset{\text{def}}{=} \{n \in \omega: f(n) \notin Y\} \in u$.
Now $\{f(n): n \in \omega\}$ is a closed discrete subset of X, and X is collectionwise normal. So we may define, recursively on ω, a family $\{H(n): n \in \omega\}$ of open sets of X satisfying

(1) $n \in \omega \Rightarrow cl(Y \cup \{f(m): m \neq n\} \cup (\cup\{H(m): m < n\})) \subseteq int(X-H(n))$, and

(2) $n \in A \Rightarrow f(n) \notin H(n)$.

Let $Z = cl(\cup\{G(n): n \in A\})$, where

$$G(n) = \begin{cases} X(n) \cap (\cup\{H(n): n \in \omega\}) & \text{if } n \in A \\ X(n) & \text{if } n \notin A. \end{cases}$$

From (1), $Z \cap Y = \emptyset$. From (2), $Z \in \Psi(u,f)$. Allowing Y to be arbitrary shows $\Psi(u,f) \in X^*$. Further, $A^* \times \nabla\langle G(n)\rangle$ is an open nbhd of $(u,E(f))$, and, from (1) and (2) we have (*a fact we use later*)

(3) $\Psi(A^* \times \nabla\langle G(n)\rangle) = \Psi(\omega^* \times \nabla\langle X(n)\rangle) - (X - \cup\{H(n): n \in \omega\})^*$

is a $\Psi(\omega^* \times \nabla X(n))$ - nbhd of $\Psi(u,f)$ missing Y^*.

Therefore, Ψ is an open map.

To see that Ψ is continuous with dense image, we suppose $Z \in \mathcal{Z}(X)$ and $Z^* \neq X^*$. Then set

$$B = \{n \in \omega: X(n) - Z \neq \emptyset\} \text{ and}$$

$$O(n) = \begin{cases} X(n) - Z & \text{if } n \in B \\ X(n) & \text{if } n \notin B. \end{cases}$$

Since each $K(n)$ is compact, $B^* \neq \emptyset$. Checking the equation

$$\Psi^{\leftarrow}(X^*-Z^*) = (B^*\times\nabla\langle O(n)\rangle) \cup ((\omega-B)^* \times \nabla\langle X(n)\rangle)$$

completes the proof.

The previous lemma was motivated by the following: *Assume* $\exists$ *a* κ-*scale*. If $X(n)$ is first countable $\forall\, n \in \omega$. Then every non-isolated point of $\nabla\langle X(n)\rangle$ has a well-ordered, by $\supseteq$, nbhd base of order-type κ [Wil], [Ro2]. So one is tempted to assume also that ω^* has a dense set of strong $P\kappa$-points and stop with that, since Ψ is an open continuous mpa. However, more effort produces a characterization.

2.4. *Definition:* Recall [Ba] that if $u \in \omega^*$, then the *ultraproduct over* u of ω many copies of ω is the quotient $\nabla_u^\omega \omega$ of ${}^\omega\omega$ (and $\nabla^\omega\omega$) obtained from the equivalence relation

$$E_u(f) = E_u(g) \text{ whenever } \{n \in \omega: f(n) = g(n)\} \in u.$$

From [Ba], $\nabla_u^\omega \omega$ is linearly ordered by

$$E_u(f) \leq E_u(g) \text{ whenever } \{n \in u: f(n) \leq g(n)\} \in u.$$

2.5. *Theorem: The following statements are equivalent:*

(1). $\exists$ *a strong* $P\omega_\alpha$*-point* u *such that* $cf(\nabla_u^\omega\omega,\leq) = \omega_\alpha$.

(2). *If* X *is a first countable, paracompact, locally compact space, then* X* *contains a dense true* η_α*-set.*

(3). $\exists$ *a strong* $P\omega_\alpha$*-point and an* ω_α*-scale.*

Proof. For simplicity we set $\kappa = \omega_\alpha$ and note that each of (1) and (2) imply κ is a regular uncountable cardinal at most $\mathfrak{c}$.

(1) $\Rightarrow$ (2). First suppose X is not σ-compact. An elementary topological fact has X as the topological sum of a collection Σ of σ-compact, non-compact spaces. If Z is a non-compact member of $\mathcal{Z}(X)$ whose compliment in X fails to have compact closure, then $\exists \mathcal{S} \in [\Sigma]^{\leq\omega}$ such that S - Z fails to have compact closure in X $\forall S \in \mathcal{S}$. Since $\cup\mathcal{S}$ is clopen in X, $(\cup\mathcal{S})^*$ is clopen in X*. Thus, $(\cup\mathcal{S})^* - Z^*$ is a nonempty open subset of X*. So it is sufficient to prove the theorem for each such $\cup\mathcal{S}$; however, $\mathcal{S}$ is σ-compact and $\cup\mathcal{S}^*$ is homeomorphic to $\beta\cup\mathcal{S} - \cup\mathcal{S}$.

Now suppose X is σ-compact. Yet another elementary fact allows us to put $X = \cup\{K(n): n \in \omega\}$, where K(n) is compact and $K(n) \subsetneq int(K(n+1))\ \forall\ n \in \omega$. We set $X(n) = K(n+1) - K(n)$. Our plan is to apply 2.3 by showing $\Psi(\omega^*\times\nabla\langle X(n)\rangle)$ has a dense set of points each of which have a well-ordered, by $\supseteq$, nbhd base of order type κ. With the plan fulfilled, the result (1) $\Rightarrow$ (2) is an immediate consequence of 2.1.

An easy argument by recursion yields $\{U_\nu: \nu < \kappa\} \subseteq u$ and $\{s_\nu: \nu < \kappa\} \subseteq {}^\omega\omega$ so that the following holds

(4). (i) $\{U^*_\nu : \nu < \kappa\}$ is a well-ordered decreasing nbhd base at u.

(ii) $\{E_u(s_\nu): \nu < \kappa\}$ is cofinal in $(\nabla^\omega_u \omega, \leq)$, and

(iii) $\mu < \nu < \kappa \Rightarrow |\{n \in U_\nu : s_\nu(n) < s_\mu(n)\}| < \omega$.

Now suppose $E(f) \in \nabla\langle int_X(X(n))\rangle$ and for each $n \in \omega$, $H(n,0) \subseteq int_X X(n)$ and $\{H(n,m): m \in \omega\}$ is a non-increasing open nbhd base at $f(n)$.

Then the set

$\{\Psi(U^* \times \nabla\langle H(n,r(n))\rangle): U \in u,\ r \in {}^\omega\omega\}$

is an open nbhd base in $\Psi(\omega^* \times \nabla\langle X(n)\rangle)$ of $\Psi(u,f)$ (see the comment in the third paragraph of the proof of 2.3). On the other hand, given $U \in u$ and $r \in {}^\omega\omega$ $\exists\ \nu < \kappa$, from (i)-(iii), such that $U^*_\nu \subseteq U^*$ and

$|\{n \in U_\nu : s_\nu(n) < r(n)\}| < \omega$.

From the definition of Ψ,

$\Psi(U^*_\nu \times \nabla\langle H(n,s_\nu(n))\rangle) \subseteq \Psi(U^* \times \nabla\langle H(n,r(n)\rangle)$.

So the desired nbhd base at $\Psi(u,f)$ is

$\{\Psi(U^*_\nu \times \nabla\langle H(n),s_\nu(n)\rangle): \nu < \kappa\}$.

Finally, A^* homeomorphic to ω^* $\forall A \in [\omega]^\omega$ implies the set

$P = \{u \in \omega^* : u$ is a strong $P\kappa$-point and $cf(\nabla^\omega_u \omega, \leq) = \kappa\}$

is dense in ω^*. Therefore, $\Psi(P \times \nabla\langle int_X(X(n))\rangle$ is the subspace we seek.

(2) $\Rightarrow$ (1). We consider a test space $X = (\Pi exp(\omega)) \times \omega$. According to (2) there is a $z \in X^*$ with a well-ordered decreasing nbhd base $\{Z^*_\nu : \nu < \kappa\}$ consisting of clopen sets (since X is 0-dimensional). Let $u \in \omega^*$ be its *trace* on ω^*; i.e.,

$u \doteq \{U \in [\omega]^\omega : (\Pi exp(\omega)) \times U \in z\}$.

Clearly, u is a strong $P\kappa$-point.

For $\nu < \kappa$ and $n \in \omega$, set $s_\nu(n) = 0$, whenever $Z_\nu \cap (\Pi exp(\omega) \times \{n\}) = \emptyset$, otherwise set $s_\nu(n)$ equal to

$$\sup\{m \in \omega: \exists i \in 2, Z_\nu \cap (\Pi exp(\omega) \times \{n\}) \subseteq \pi_m^{\leftarrow}(i) \times \{n\}\}.$$

Since Z_ν is clopen (or at least we can assume it is) in X, $s_\nu \in {}^\omega\omega$. Now if $\mu < \nu$, then $Z_\nu - Z_\mu$ is compact. So

$$|\{n \in \omega: s_\nu(n) < s_\mu(n)\}| < \omega.$$

Therefore, $\{E_u(s_\nu): \nu < \kappa\}$ is a non-decreasing subset of $(\nabla_u^\omega\omega, \leq)$.

For $r \in {}^\omega\omega$ and $i \in 2$, we set

$$Z(i) = \cup\{\pi_{r(n)+1}^{\leftarrow}(i) \times \{n\}: n \in \omega\}$$

Since $X = Z(0) \cup Z(1)$, $\exists j \in 2$ with $Z(j) \in z$. So we find $\nu < \kappa$ with $Z_\nu^* \subseteq Z(j)^*$. Since $Z_\nu - Z(j)$ is compact,

$$\{n \in \omega: Z_\nu \cap (\Pi exp(\omega) \times \{n\}) \neq \emptyset\}^* = \{n \in \omega: r(n) + 1 \leq s_\nu(n)\}^*.$$

Therefore, $E_u(r) < E_u(s_\nu)$.

(3) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (3). As we have in (1) $\Rightarrow$ (2) obtain $\{U_\nu: \nu < \kappa\} \subseteq u$ and $\{s_\nu: \nu < \kappa\} \subseteq {}^\omega\omega$ satisfying (4). Fix $\nu < \kappa$ and let $\{a_n: n \in \omega\}$ be the natural ordering of U_ν. We define $r_\nu \in {}^\omega\omega$ by $r_\nu(n) = s_\nu(a_n)$. We show $\{r_\nu: \nu < \kappa\}$ is a κ-scale.

Suppose that $\mu < \nu < \kappa$. Then (4)(iii) implies there is an $a \in \omega$ such that $m \in U_\mu$ and $s_\mu(m) < s_\nu(m)$ whenever $m \geq a$ and $m \in U_\nu$. So if $a_n \geq a$, then $r_\mu(n) = s_\mu(a_n) < s_\nu(a_n) = r_\nu(n)$. Thus, $\{E(r_\nu): \nu < \kappa\}$ is well-ordered in $\nabla^\omega\omega$.

Suppose that $r \in {}^\omega\omega$. Then (4)(i) and (ii) imply there is an $a \in \omega$ and there are $\mu < \nu < \kappa$ such that

$\Sigma_{m=o}^{n} r(m) < S_{\mu}(n)$ whenever $n \geq a$ and $n \in U_{\nu}$. So $a_n \geq a$ implies that $r(n) < s_{\mu}(a_n) < r_{\nu}(n)$.

2.6. *Corollary* [Sm]: *Suppose ω^* is not the union of at most ¢ nowhere dense subsets. Then ω^* has a dense orderable subspace.*

Proof. The supposition implies (1). $\exists$ a strong P¢-point (which, by itself, is sufficient) and (2). $\exists$ a ¢=scale. ((1) is due to Hechler, unpublished. (2) is in [BPS].)

2.7. *Remarks:* (1). J. van Mill informs me that while studying an example space of the form $K \times \omega$, where K is compact, he discovered (unpublished) the map Ψ of 2.3. However, he did not know the generality of 2.3 nor that, in his case, the map Ψ is open.

Also prior to our work, E. K. van Douwen (unpublished) discovered that a quotient of Ψ restricted to ultraproducts (see 2.4 and [Ba]) becomes an embedding. He applies this map to produce non-homeomorphic Parovičenko spaces possessing dense orderable subspaces (also see [vDvM] and [Wi3]).

(2). The use of κ-scales to study Čech-Stone remainders is not new, see [Co] for example. The idea of replacing κ-scales with "ultraproduct scales," in 2.4 and 2.5, came from our proof (unpublished) that ¢ $\leq \omega_2$ implies $\square^{\omega}\omega+1$ is paracompact (see [Ro2] and [Wi1]).

(3). For the general theorem characterizing "χ has a dense orderable subspace" see [Wi4].

(4). The "other" theorem of [CN1] "If X is a realcompact, locally compact, non-compact space of weight at most $\mathfrak{c}$, then $(X^*)_{\mathfrak{c}}$ and $(\Pi exp(\mathfrak{c}))_{\mathfrak{c}}$ are homeomorphic" assumes CH but remains true in the models provided in 1.5 and 1.7(2). On the other hand, it is clearly false if $\exists$ *a strong* P_κ*-point* and $\kappa < \mathfrak{c}$. 3.10 shows this, as well as $\exists$ *a* κ*-scale*, is consistent.

3. Embedding $(\Pi \exp(\kappa))_\kappa$

The singular motivation for this entire paper is the theorem of W. W. Comfort and S. Negrepontis [CN1] stated as (1) $\Rightarrow$ (2) below.

3.1. *Theorem: The following statements are equivalent:*

(1). CH.

(2). *If* X *is a realcompact, locally compact, non-compact space of weight at most* $\mathfrak{c}$, *then the* (dense) *subspace of* P*-points of* X^* *is homeomorphic to* $(\Pi exp(\mathfrak{c}))_{\mathfrak{c}}$.

(3). *If* X *is a separable,* σ*-compact, locally compact, non-compact space, then* X^* *has a dense orderable subspace.*

Proof. (2) $\Rightarrow$ (3). A σ-compact space is realcompact and a separable space has at most $\mathfrak{c}$ regular-open sets.

(3) $\Rightarrow$ (1). If $\omega_2 \leq \mathfrak{c}$, then $\mathcal{D}(\omega_2)^*$ has no dense orderable subspace (see 1.6).

Indeed, after we obtained our first results, the entirety of §1, we set out to obtain the strongest possible approximation of 3.1(2) in the presence of $\omega_1 < \mathfrak{c}$. Of course, 3.1((3) $\Rightarrow$ (1)) says the class of separable spaces is too large; in fact, if there is no strong P-point, then $\mathcal{D}(\omega_1)^*$

rules out the spaces with point-character at most ω_1. On the other hand, if there is a strong P-point, then $((\omega_2+1)\times\omega)^*$ has a P-point which is a not a strong Pκ-point for any κ. Yet another bound, for spaces with countable π-weight, is provided by $((\beta\omega)\times\omega)^*$. Therefore, the authors of [HN] presented a strong approximation with *assume* MA. *If* X *is a locally compact, non-compact metrizable space of weight at most* ¢, *then the subspace of* P¢-*points of* X* *is homeomorphic to* $(\Pi exp(¢))_{¢}$. For the remainder of this section we present other approximations. Our major result is

3.2. *Theorem: The following statements are mutually consistent with the axioms of* ZFC*:*

(1). $\omega_1 < ¢$.

(2). *If* X *is a locally compact, non-compact metrizable space of weight at most* ¢, *then* $(\Pi exp(\omega_1))_{\omega_1}$ *is densely embedded in* X*.

(3). *If* X *is a paracompact, locally compact, non-compact space of weight at most* ¢, *and if* $\chi(x,X) \leq \omega_1\ \forall x \in X$, *then the canonical* η_1-*set is densely embedded in* X*.

We present in [Wi5] a model where (1), (2), and (3) hold; however, here we find conditions which simplify the search for such a model. First, one should observe that a reduction to the class of σ-compact spaces is made possible by the argument in the first paragraph of 2.5(1) $\Rightarrow$ (2) along with the following lemma due to Hung and Negrepontis.

3.3. *Lemma* [CN2, 15.3]*: Suppose* κ *is a regular uncountable non-weakly compact cardinal. A space* Y *is* *homeomorphic to* $(\Pi exp(\kappa))_\kappa$ *iff each of the following are satisfied*

(1). *every element of* Y *has a well-ordered* nbhd *base of order-type* κ.

(2). $\forall \nu < \kappa\ \exists\ A_\nu$ *an open partition of* Y.

(3). *If* $A = \cup\{A_\nu : \nu < \kappa\}$, *then*

(i) *every filter in* A *has non-empty intersection.*

(ii) $\mathcal{G} \in [A]^{<\kappa} \Rightarrow \cap\mathcal{G} \in A$.

(iii) $|A| \leq 2^\kappa$ *and* A *is a subbase for* Y.

From 3.3, the following is a direct consequence of an equivalence proved in [vD1, 6.3].

3.4. *Lemma: Suppose* $(\Pi exp(\kappa))_\kappa$ *is densely embedded in* ω^*. *If* X *is a realcompact, locally compact, non-compact space of countable* π*-weight, then the intersection of less than* κ *many non-empty open subsets of* X* *has non-empty interior.*

3.5. *Definition:* For a regular uncountable cardinal $\kappa \leq ¢$, $\#(\kappa)$ means the following statement is true: $\exists$L *a dense subspace of* ω^* *and* $\exists\{s_\nu : \nu < \kappa\} \subseteq {}^\omega\omega$ *such that* L *is homeomorphic to* $(\Pi exp(\kappa))_\kappa$ *and* $\{E_u(s_\nu) : \nu < \kappa\}$ *is an increasing sequence cofinal in* $(\nabla_u^\omega\omega, \leq)$ $\forall u \in$ L.

Observe that MA implies $\#(¢)$.

3.6. *Lemma: Assume* $\#(\kappa)$. *If* X *is a* σ*-compact, locally compact, non-compact metrizable space, then* $(\Pi exp(\kappa))_\kappa$ *is densely embedded in* X*.

Proof. Following 2.5 we write X as the disjoint union of subspaces $X(n)$, for $n \in \omega$, where $cl(X(n)) \subseteq cl(int(X(n+1)))$ is compact $\forall n \in \omega$. Observe that for a given $r \in {}^{\omega}\omega$, $\nabla\langle X(n)\rangle$ has a π-base $\mathcal{B}r$ consisting of all sets of the form $\nabla\langle B(n)\rangle$, where $cl_X(B(n)) \subseteq X(n)$ and $B(n)$ is an X-open ball of radius at most $1/r(n)$ $\forall n \in \omega$.

Identify L with $(\Pi exp(\kappa))_{\kappa}$. For $\nu < \kappa$ and $t \in exp(\nu)$ we set

$$R(t) = int_{\omega^*}(cl_{\omega^*}(\{u \in L: t = u|\nu\})).$$

Since $L \subseteq \cup\{R(t): t \in exp(\nu)\}$ $\forall\nu < \kappa$, we apply 3.4 to construct, recursively $\forall\nu < \kappa$, families $\mathcal{R}_\nu$ and $\mathcal{A}_\nu$ subject to the restrictions:

(1). $\mathcal{R}_\nu$ consists of pairwise-disjoint clopen sets of ω^* and $cl(\cup\mathcal{R}_\nu) = \omega^*$,

(2). $R^* \in \mathcal{R}_\nu \Rightarrow \exists t \in exp(\nu)$ with $R^* \subseteq R(t)$.

(3). $\mathcal{A}_\nu$ consists of pairwise-disjoint sets of X^* having form $\Psi(R^* \times \nabla\langle B(n)\rangle)$ where $R^* \in \mathcal{R}_\nu$ and $\nabla\langle B(n)\rangle \in \mathcal{B}s_\nu$,

(4). $R^* \in \mathcal{R}_\nu \Rightarrow \Psi(R^* \times \langle X(n)\rangle) \subseteq cl_{X^*}(\{\Psi(R^* \times \langle B(n)\rangle) \in \mathcal{A}_\nu\})$,

(5). $\Psi(R^* \times \langle B(n)\rangle) \in \mathcal{A}_\nu \Rightarrow \exists\Psi(R^*_\mu \times \langle B_\mu(n)\rangle) \in \mathcal{A}_\mu$ $\forall\mu < \nu$ with $(\cup\{cl(B(n)): n \in R\})^* \subseteq \cap\{int_{X^*}((\cup\{B_\nu(n): n \in R_\nu\})^*): \mu < \nu\}$.

If we are careful to choose the right representatives R and B(n), then the construction is simple. Thus we omit the details. Let $\mathcal{A} = \cup\{\mathcal{A}_\nu: \nu < \kappa\}$.

Suppose M is a maximal filter in the poset $(\mathcal{A},\subseteq)$, then (3) implies M picks at most one member $\Psi(R^*_\nu \times \nabla\langle B_\nu(n)\rangle)$ from any given $\mathcal{A}_\nu$. On the other hand, (4) and (5) imply M picks at least one member from each $\mathcal{A}_\nu$. For each $\nu < \kappa$ find

$t_\nu \in exp(\nu)$ with $R^*_\nu \subseteq R(t_\nu)$. Since L is dense in ω^*,

$$\mu<\nu<\kappa \Rightarrow R(t_\mu) \cap R(t_\nu) \neq \emptyset \Rightarrow t_\mu \subseteq t_\nu.$$

So $t = \cup\{t_\nu: \nu < \kappa\} \in L$. Further, since $\{R(t_\nu): \nu < \kappa\}$ is a nbhd base for t and since $\cap\{R^*_\nu: \nu < \kappa\}$ is compact, $\{R^*_\nu: \nu < \kappa\}$ is a nbhd base for t. Set

$$z_M = \{Z \in \mathcal{Z}(X): \{n \in R_\nu: cl(B_\nu(n)) \cap Z \neq \emptyset\} \in t \ \forall\nu < \kappa\}.$$

We show $z_M \in X^*$.

Since $t \in \omega^*$, z_M contains no compact members of $\mathcal{Z}(X)$. Thus it is sufficient to prove $W \cap Z \in z_M$ whenever $W, Z \in z_M$. Suppose, by way of contradiction, $W \cap Z \notin z_M$. Then $\exists\lambda < \kappa$ such that $T \in t$, where $n \in T$ iff $cl(B_\lambda(n)) \cap W \neq \emptyset$, $cl(B_\lambda(n)) \cap Z \neq \emptyset$, and $cl(B_\lambda(n)) \cap W \cap Z = \emptyset$. Since X(n) has compact closure $\forall n \in \omega$, we may define $d \in {}^\omega\omega$ by $d(n) = 1$ if $n \notin T$, and d(n) is the distance between $cl(B_\lambda(n)) \cap W$ and $cl(B_\lambda(n)) \cap Z$ if $n \in T$. From the assumption $\#(\kappa)$ $\exists\mu,\nu$, $\lambda<\mu<\nu<\kappa$ such that $R^*_\mu \subseteq R^*_\lambda \cap T^*$ and

$$|\{n \in R_\nu: d(n) \leq 2/s_\mu(n)\}| < \omega.$$

From (3), $W, Z \in z_M$ implies, on the other hand

$$\{n \in \omega: d(n) \leq 2/s_\mu(n)\} \in t.$$

As this is absurd, $W \cap Z \in z_M$ whenever $W, Z \in z_M$. Thus, $z_M \in X^*$.

To complete the proof, set

$$D = \{z_M: M \text{ is a maximal filter in } (\mathcal{A},\subseteq)\}.$$

3.3 and $\{D \cap int_{X^*}(cl_{X^*}(A)): A \in \mathcal{A}\}$ show D and $(\Pi exp(\kappa))_\kappa$ are homeomorphic.

3.7. *Definition:* For a cardinal κ, an object familiar to set-theorists is the *complete binary tree of height* κ. It is the poset

$$\mathrm{TREE}(\kappa) = (\cup\{exp(\nu): \nu < \kappa\}, \subseteq).$$

Obviously, a space Y has a π-base $\mathcal{B}$ (or rather $(\mathcal{B},\supseteq)$) isomorphic to $\mathrm{TREE}(\kappa)$ whenever $(\Pi(exp(\kappa)))_\kappa$ is densely embedded in Y.

3.8. *Lemma* [Wi2]: *ω^* has a π-base isomorphic to* $\mathrm{TREE}(\omega_1)$ *iff* X^* *has a π-base isomorphic to* $\mathrm{TREE}(\omega_1)$ *for each realcompact, locally compact, non-compact space* X *of π-weight at most* ¢.

3.9. *Lemma: Assume* $\#(\omega_1)$. *If* X *is a σ-compact, locally compact, non-compact space of π-weight at most* ¢, *and if* $\chi(x,X) \le \omega_1$ $\forall x \in X$, *then the canonical η_1-set is densely embedded in* X^*.

Proof. Once again write X as a disjoint union $\{X(n): n \in \omega\}$, where each X(n) has compact closure. We first show every point of $\Psi(L\times\nabla\langle int_X(X(n))\rangle$ has a well-ordered, by $\supseteq$, nbhd base of order type ω_1.

Let $(u,E(f)) \in L\times\nabla\langle int_X(X(n))\rangle$. From 2.5 we have the desired result if

$$\{n \in \omega: \chi(f(n),X(n)) \le \omega\} \in u.$$

So we assume instead that

$$U = \{n \in \omega: \chi(f(n),X(n)) = \omega_1\} \in u.$$

For each $n \in U$ fix an open nbhd base $\{H(n,\nu): \nu < \omega_1\}$ of f(n) in $int_X(X(n))$. For $\nu < \omega_1$ fix a bijection $b_\nu \in {}^\nu U$. Now, given $\lambda < \nu$ and $b_\nu(\lambda) \le m \in U$, we find

$$\cap\{H(m,\mu): b_\nu(\mu) \le m\} \subseteq H(m,\lambda).$$

So we set

$$G(\nu) = (\cup\{\cap\{H(n,\mu): b_\nu(\mu) \le n\}: n \in U\})^*.$$

Then $\Psi(u,f) \in int_{X^*}(G(\nu))\ \forall\nu < \omega_1$. Further, $\lambda<\mu<\kappa$ implies

$$G(\nu) \cap \Psi(L\times\nabla\langle int_X(X(n))\rangle) \supseteq G(\lambda) \cap \Psi(L\times\nabla\langle int_X(X(n))\rangle)$$

and

$$G(\nu) \subseteq \cap\{(\cap\{H(n,\mu): n \in U\})^*: \mu < \nu\}.$$

From the second paragraph of the proof of 2.3, we see that the desired nbhd base is $\{G(\nu): \nu < \omega_1\}$.

Now of course we chose the $X(n)$ so that

$$X = cl(\cup\{int(X(n)): n \in \omega\}).$$

Thus, the proof of 2.5 (via 2.1) shows $\Psi(L\times\nabla\langle int(X(n))\rangle)$ contains a dense true η_1-set. In the same way one proves a first countable space with a countable π-base and no isolated points contains a dense copy of the rationals, a simple modification in the Cantor-Hausdorff η_α-set theorem [Si, 465-468], shows that a true η_α-set with a π-base isomorphic to TREE(ω_α) contains a dense copy of the canonical η_α-set.

3.10. *Lemma: Suppose* $\mathcal{M}$ *is a* ctm ZFC *and* $\mathcal{M} \models \omega_1 < \cent$ *and* κ *is a regular uncountable cardinal at most* $\cent$. *Then there is a generic extension* $\mathcal{N}$ *of* $\mathcal{M}$ *such that* $\mathcal{N} \models$

(1). $\cent^{\mathcal{M}} = \cent^{\mathcal{N}}$.

(2). $\exists$ *a* κ*-scale.*

(3). ω^* *has a dense subspace homeomorphic to* $(\Pi exp(\kappa))_\kappa$.

Proof (sketch, the details will appear in [Wi5]). $\mathcal{N}$ is constructed as the direct limit of a sequence $\{\mathcal{M}_\nu: \nu < \kappa\}$ of generic extensions of $\mathcal{M}$ where $\mu<\nu<\kappa \Rightarrow \mathcal{M}_\mu \subseteq \mathcal{M}_\nu$. At successor stages we use a c.c.c. poset $\mathcal{P}$ to

(i) add an $s_\nu \in \mathcal{M}_\nu \cap {}^\omega\omega$ such that $E(r) \leq E(s_\nu)$ $\forall r \in \mathcal{M}_{\nu-1} \cap {}^\omega\omega$.

(ii) add an $\mathcal{A}_\nu \in \mathcal{M}_\nu$, $\mathcal{A}_\nu \subseteq [\omega]^\omega$, $|\mathcal{A}_\nu| = \mathfrak{c}^{\mathcal{M}}$ such that $\{u \in \mathcal{M}_{\nu-1} \cap \omega^*: A \in \mathcal{A}_\nu, |A-U| < \omega\ U \in u\}$ is dense in $\mathcal{M}_{\nu-1} \cap \omega^*$.

$\mathcal{P}$ is found as follows. First choose a dense set D of $\omega^* \cap \mathcal{M}_{\nu-1}$ with $|D| = \mathfrak{c}$. The elements of $\mathcal{P}$ will be all functions ϕ with $dom(\phi) \in [D]^{<\omega}$ such that for each $u \in dom(\phi)$, $\phi(u) = (U,r,f)$ where $U \in u$, $r \in ({}^\omega\omega \cap \mathcal{M}_{\nu-1})$, and where f is a function with $dom(f) \in [\omega]^{<\omega}$, $ran(f) \subseteq \omega$, and $r(n) \leq f(n)$ $\forall n \in dom(f)$. $\mathcal{P}$ is to order by $\phi \leq \psi$ if $dom(\psi) \subseteq dom(\phi)$, and if $u \in dom(\psi)$ has $\psi(u) = (V,s,g)$ and $\phi(u) = (U,r,f)$, then

(a). $U \subseteq V$.

(b). $s(n) \leq r(n)$ $\forall n \in \omega$.

(c). $dom(g) \subseteq dom(f)$.

(d). $n \in dom(f) - dom(g) \Rightarrow n \in V-U$ and $s(n) \leq f(n)$.

The result follows from standard iterated forcing facts (see [Ro1], [Ro2]).

Proof of 3.2. In 3.10 let $\mathcal{M} \models MA + \mathfrak{c} = \omega_2$, and take $\kappa = \omega_1$. Then $\#(\omega_1)$ is true. Now apply 3.6 and 3.9.

3.11. *Remarks:* (1). Ideally, I would have liked in 3.2(2) to say "the subspace of all strong $P\omega_1$-points of X^* is homeomorphic to $(\Pi exp(\omega_1))\omega_1$." For that, however, I need to release $\omega_1 = \omega_1^{\omega_1}$ from the hypothesis of [CN], 15.11(a)].

(2). A theorem in [BPS] together with a theorem in [S1] shows that when $(\Pi exp(\kappa))_\kappa$ is densely embedded in ω^*, then κ is the least cardinal of an unbounded family in $(\nabla^\omega_u\omega,\leq)$. The latter is awfully close to $cf(\nabla^\omega_u\omega,\leq) = \kappa$ for some $u \in \omega^*$; therefore, the second condition in the definition of $\#(\kappa)$ may be superfluous.

(3). It is conceivable that the dense subspace D of X^* found in 3.6 is disjoint from $\Psi(\omega^* \times \nabla\langle X(n)\rangle)$ whenever X has no isolated points. In fact, applying the techniques of [CS] we can make D to consist, in this case, entirely of remote points of X.

(4). 3.9 contains the proof that $\nabla\langle X(n)\rangle$ consists of P-points of character ω_1 whenever $\chi(x, X(n)) = \omega_1$ $\forall x \in X(n)$ $\forall n \in \omega$. We first discovered this fact while writing up [Wi1].

(5). It appears that in every model presented in this paper, ω^* and X^* are co-absolute whenever X is a first countable, paracompact, locally compact, non-compact space of π-weight at most $\mathfrak{c}$. However, the question: Are ω^* and $\mathbb{R}^*$ co-absolute, in ZFC, remains open. See [Wo] and [Wi2].

References

BPS B. Balcar, J. Pelant and P. Simon, *The space of ultrafilters on* N *covered by nowhere dense sets*, Fund. Math. 110 (1980), 11-24.

Ba P. Bankston, *Ultraproducts in topology*, Gen. Top. and Appl. 7 (1977), 283-308.

Bu B. Burgess, *Forcing*, Handbook of Mathematical Logic, North-Holland (1977), 403-452.

CS S. Chae and J. Smith, *Remote points and* G-*spaces*, Top. and Appl. 11 (1980), 243-246.

Co W. W. Comfort, *Compactifications: Recent results from several countries*, Topology Proc. 2 (1977), 61-80.

CN1 ________ and S. Negrepontis, *Homeomorphs of three subspaces of* βN-N, Math. Zeit. 107 (1968), 53-58.

CN2 ________, *The theory of ultrafilters*, Die Grund. der Math. Wiss. in Einz. 211, Springer-Verlag (1974).

vD1 E. van Douwen, *Transfer of information about* βN-N *via open remainder maps* (to appear).

vD2 ________, *Covering and separation in box products*, Surveys in General Topology, Academic Press (1980), 55-130.

vDvM ________ and J. van Mill, *Parovičenko's characterization of* $\beta\omega-\omega$ *implies* CH, Proc. A.M.S. 72 (1978), 539-541.

He S. Hechler, *On the existence of certain cofinal subsets of* ${}^{\omega}\omega$, Proc. Symposia in Pure Math. 13 II (1974), 155-173.

HN H. Hung and S. Negrepontis, *Spaces homeomorphic to* $(2^{\alpha})_{\alpha}$ II., Trans. A.M.S. 188 (1974), 1-30.

Ku K. Kunen, *Some points in* βN, Proc. Cambridge Phil. Soc. (1976), 385-398.

Pa I. Parovičenko, *A universal bicompact of weight* $\aleph_1$, Sov. Math. Dokl. 4 (1963), 592-595.

Ro1 J. Roitman, P-*points in iterated forcing extensions*, Proc. A.M.S. 69 (1978), 314-318.

Ro2 ________, *More paracompact box products*, Proc. A.M.S. 74 (1979), 171-176.

Ru M. Rudin, *Lectures in set-theoretic topology*, Conf. Bd. Math. Sci. Reg. Conf. Ser. 23, A.M.S. (1977).

Si W. Sierpiński, *Cardinal and ordinal numbers*, Monographs in Math. 34, PWN (1965).

Sm P. Simon, *A somewhat surprising subspace of* βN-N, Comm. Math. Univ. Carol. 19 (1978), 383-388.

So R. Solovay, *A model of set theory in which every set of reals is Lebesque measurable*, Annals of Math. 92 (1970), 1-56.

Sl R. Solomon, *Families of sets and functions*, Czech. Math. J. 27 (102) (1977), 556-559.

TZ G. Takeuti and W. Zaring, *Axiomatic set theory*, Springer-Verlag (1973).

Wil S. Williams, *Is* $\square^{\omega}(\omega+1)$ *paracompact*, Topology Proc. 1 (1976), 141-146.

Wi2 ________, *Trees, Gleason spaces, and co-absolutes of* $\beta N-N$, Trans. A.M.S. 271 (1982), 83-100.

Wi3 ________, *Some compact* F-*spaces of continuum weight*, Topology and Order Structures II, Math. Centre Tracts (to appear).

Wi4 ________, *Spaces with dense orderable subspaces*, Topology and Order Structures I, Math. Centre Tracts (1981), 27-49.

Wi5 ________, *A simple forcing exercise* (manuscript).

Wm E. Wimmers, *The Shelah* P-*point independence theorem* (to appear).

Wo R. G. Woods, *Co-absolutes of remainders of Stone-Čech compactifications*, Pac. J. Math. 37 (1971), 545-560.

Institute for Medicine and Mathematics

Athens, Ohio 45701

and

The State University of New York at Buffalo

Buffalo, New York 14214