46 Injective modules

Recall. If \( R \) is a ring with identity then an \( R \)-module \( P \) is projective iff one of the following equivalent conditions holds:

1) For any homomorphism \( f: P \to N \) and an epimorphism \( g: M \to N \) there is a homomorphism \( h: P \to M \) such that the following diagram commutes:

\[
\begin{array}{ccc}
P & \xrightarrow{h} & M \\
\downarrow{f} & & \downarrow{g} \\
N & & \\
\end{array}
\]

2) Every short exact sequence \( 0 \to N \xrightarrow{f} M \xrightarrow{g} P \to 0 \) splits.

46.1 Proposition. Let \( R \) be a ring and let \( J \) be an \( R \)-module. The following conditions are equivalent.

1) For any homomorphism \( f: N \to J \) and an monomorphism \( g: M \to N \) there is a homomorphism \( h: M \to J \) such that the following diagram commutes:

\[
\begin{array}{ccc}
J & \xrightarrow{h} & N \\
\downarrow{f} & & \downarrow{g} \\
M & & \\
\end{array}
\]

2) Every short exact sequence \( 0 \to J \xrightarrow{f} M \xrightarrow{g} N \to 0 \) splits.

Proof. Exercise. \( \square \)
46.2 Definition. An $R$-module $J$ is an injective module if $J$ satisfies one of the equivalent conditions of Proposition 46.1.

46.3 Theorem (Baer’s Criterion).
Let $R$ be a ring with identity and let $J$ be an $R$-module. The following conditions are equivalent.

1) $J$ is an injective module.

2) For every left ideal $I \triangleleft R$ and for every homomorphisms of $R$-modules $f : I \rightarrow J$ there is a homomorphism $\bar{f} : R \rightarrow J$ such $f|_I = \bar{f}$.

Proof.
1) $\Rightarrow$ 2) Given a homomorphism $f : I \rightarrow J$ we have a diagram

\[ \begin{array}{ccc}
J & \rightarrow & J \\
\downarrow & f & \downarrow \\
I & \leftarrow & R
\end{array} \]

where $i : I \rightarrow R$ is the inclusion homomorphism. By the definition of an injective module there is a homomorphism $\bar{f} : R \rightarrow J$ such that $f = \bar{f}i = \bar{f}|_I$.

2) $\Leftarrow$ 1) Assume that $J$ is an $R$-module satisfying 2). It is enough to show that if $M$ is an $R$-module, $N$ is a submodule of $M$, and $f : N \rightarrow J$ is an $R$-module homomorphism then there exists a homomorphism $\bar{f} : M \rightarrow J$ such that $\bar{f}|_N = f$.

Let $S$ be a set of all pairs $(K, f_K)$ such that

(i) $K$ is a submodule of $M$ such that $N \subseteq K \subseteq M$

(ii) $f_K : K \rightarrow J$ is a homomorphism such that $f_K|_N = f$.
Define partial ordering on $S$ as follows:

$$(K, f_K) \leq (K', f_{K'}) \text{ if } K \subseteq K' \text{ and } f_{K'}|_K = f_K$$

Check: assumptions of Zorn’s Lemma 29.10 are satisfied in $S$, and so $S$ contains a maximal element $(K_0, f_{K_0})$.

It will suffice to show that $K_0 = M$. Assume, by contradiction, that $K_0 \neq M$, and let $m_0 \in M - K_0$. Define

$$I := \{r \in R \mid rm_0 \in K_0\}$$

Check: $I$ is an ideal of $R$ and the map

$$g: I \rightarrow J, \quad g(r) = f_{K_0}(rm_0)$$

is a homomorphism of $R$-modules. By the assumptions on $J$ we have a homomorphism $\bar{g}: R \rightarrow J$ such that $\bar{g}|_I = g$. Define

$$K_0 + Rm_0 := \{k + rm_0 \mid k \in K, \ r \in R\}$$

Check: $K_0 + Rm_0$ is a submodule of $M$ and the map

$$f': K_0 + Rm_0 \rightarrow J, \quad f'(k + rm_0) = f_{K_0}(k) + \bar{g}(r)$$

is a well defined homomorphism of $R$-modules such that $f'|_N = f$. This shows that $(K_0 + Rm_0, f') \in S$. We also have

$$(K_0, f_{K_0}) < (K_0 + Rm_0, f')$$

This is impossible since by assumption $(K_0, f_{K_0})$ is a maximal element in $S$.  

46.4 Corollary. Let $R$ be an integral domain and let $K$ the field of fractions of $R$. Then $K$ is an injective $R$-module.

Proof. Let $I$ be an ideal of $R$ and let $f: I \rightarrow K$ be a homomorphism of $R$-modules. For $0 \neq r, s \in I$ we have

$$rf(s) = f(rs) = sf(r)$$
As consequence in $K$ we have $f(r)/r = f(s)/s$ for any $0 \neq r, s \in I$. Denote this element by $a$. Define

$$\bar{f} : R \to K, \quad \bar{f}(r) := ra$$

Check: $\bar{f}$ is a homomorphism of $R$-modules and $\bar{f}|_I = f$.

By Baer's Criterion (46.3) it follows that $K$ is an injective $R$-module.

46.5 Example. $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

46.6 Definition. Let $R$ be an integral domain. An $R$-module $M$ is divisible if for every $r \in R - \{0\}$ and for every $m \in M$ there is $n \in M$ such that $rn = m$.

46.7 Theorem. If $R$ is a PID then an $R$-module $J$ is injective iff $J$ is divisible.

Proof. Exercise.

46.8 Example.

Since $\mathbb{Z}$ is a PID injective $\mathbb{Z}$-modules are divisible $\mathbb{Z}$-modules (i.e. divisible abelian groups).

Exercise: an abelian group $G$ is divisible iff $G$ is isomorphic to a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ for various primes $p$.

46.9 Corollary. If $R$ is a PID, $J$ is an injective $R$-module and $K$ is a submodule of $J$ then $J/K$ is injective.
Proof. Since $J$ is divisible thus so is $J/K$ (check!). \hfill \Box

46.10 Note. If $R$ is not a PID then a quotient of an injective $R$-module need not be injective.

Note. If $R$ is a ring with identity then for any $R$-module $M$ there exists an epimorphism of $R$-modules:

$$f: P \longrightarrow M$$

where $P$ is a projective module (take e.g. $P = \bigoplus_{m \in M} R$).

46.11 Theorem. If $R$ is a ring with identity then for any $R$-module $M$ there exist a monomorphism

$$j: M \longrightarrow J$$

where $J$ is an injective $R$-module.

46.12 Lemma. For any abelian group $G$ there exists a monomorphism

$$i: G \longrightarrow D$$

where $H$ is a divisible abelian group.

Proof. We have an epimorphism $f: \bigoplus_{g \in G} \mathbb{Z} \rightarrow G$ which gives an isomorphism

$$\varphi: G \xrightarrow{\cong} \bigoplus_{g \in G} \mathbb{Z}/\text{Ker}(f)$$

Moreover, the monomorphism $\bigoplus_{g \in G} \mathbb{Z} \rightarrow \bigoplus_{g \in G} \mathbb{Q}$ induces a monomorphism

$$\psi: \bigoplus_{g \in G} \mathbb{Z}/\text{Ker}(f) \longrightarrow \bigoplus_{g \in G} \mathbb{Q}/\text{Ker}(f)$$

We can take $D := \bigoplus_{g \in G} \mathbb{Q}/\text{Ker}(f)$ and $i := \psi \varphi$. \hfill \Box

189
46.13 Note. Let $G$ is an abelian group, let $R$ let be a ring, and let $\text{Hom}_\mathbb{Z}(R, G)$ be the set of all homomorphisms of abelian groups $\varphi: R \to G$.

Check: $\text{Hom}_\mathbb{Z}(R, G)$ is an $R$-module with pointwise addition and with multiplication by elements of $R$ given by

$$(r \cdot \varphi)(s) := \varphi(sr)$$

for $r, s \in R$.

46.14 Lemma. If $D$ is a divisible abelian group and $R$ is a ring with identity then $\text{Hom}_\mathbb{Z}(R, D)$ in an injective $R$-module.

Proof. Exercise. □

Proof of Theorem 46.11. Let $M$ be an $R$-module. Consider $M$ as an abelian group. By Lemma 46.12 we have a monomorphism of abelian groups

$$i: M \longrightarrow D$$

where $D$ is a divisible abelian group. Consider the induced map

$$i_*: \text{Hom}_\mathbb{Z}(R, M) \rightarrow \text{Hom}_\mathbb{Z}(R, D), \quad i_*(\varphi) = i \circ \varphi$$

Check:

1) $i_*$ is a monomorphism.

2) $i_*$ is a homomorphism of $R$-modules.

Since $M$ is an $R$-module we also have a map

$$f: M \rightarrow \text{Hom}_\mathbb{Z}(R, M), \quad f(m)(r) = rm$$

Check:

1) $f$ is a monomorphism.
2) $f$ is a homomorphism of $R$-modules.

As a consequence we obtain a monomorphism of $R$-modules

$$i_* f: M \rightarrow \text{Hom}_Z(R, D)$$

Moreover, by Lemma 46.12 $\text{Hom}_Z(R, D)$ is an injective $R$-module.

□
47  Exact functors

47.1 Definition. A chain complex of $R$-modules is a sequence of $R$-modules and $R$-homomorphisms

$$
\ldots \longrightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} M_{i-2} \longrightarrow \ldots
$$

such that $d_id_{i+1} = 0$ for all $i$.

47.2 Note. If $M_* = (M_i, d_i)$ is a chain complex then $\text{Im}(d_{i+1}) \subseteq \text{Ker}(d_i)$.

47.3 Definition. If $M_* = (M_i, d_i)$ is a chain complex of $R$-modules then the $i$-th homology module of $M_*$ is the module

$$H_i(M_*) = \text{Ker}(d_i)/\text{Im}(d_{i+1})$$

Recall. A sequence

$$M_* = (\ldots \longrightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} M_{i-2} \longrightarrow \ldots)$$

is exact if $\text{Ker}(d_i) = \text{Im}(d_{i+1})$ for all $i$. Therefore $M_*$ is exact iff $H_i(M_*) = 0$ for all $i$.

47.4 Definition. Let $R, S$ be rings. A functor $F : R\text{-Mod} \rightarrow S\text{-Mod}$ is exact if for every short exact sequence of $R$-modules

$$
0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} K \longrightarrow 0
$$

the sequence

$$
0 \longrightarrow F(N) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(K) \longrightarrow 0
$$

is short exact.
47.5 Note. If $F: R	ext{-Mod} \to S	ext{-Mod}$ is a functor such that $F(0) = 0$ and

$$M_\ast = (\ldots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \to \ldots)$$

is a chain complex of $R$-modules then

$$F(M_\ast) = (\ldots \to F(M_{i+1}) \xrightarrow{F(d_{i+1})} F(M_i) \xrightarrow{F(d_i)} F(M_{i-1}) \to \ldots)$$

is a chain complex of $S$-modules.

Moreover, the functor $F$ is exact iff for every chain complex $M_\ast$ we have isomorphisms

$$F(H_i(M_\ast)) \simeq H_i(F(M_\ast))$$

for all $i$.

47.6 Note. For a ring $R$ and $R$-modules $L, M$ let $\text{Hom}_R(L, M)$ be the set of all $R$-module homomorphisms $\varphi: L \to M$. Notice that $\text{Hom}_R(L, M)$ is an abelian group (with respect to the pointwise addition of homomorphisms). Moreover, for any homomorphism of $R$-modules $f: M \to N$ the map

$$f_*: \text{Hom}_R(L, M) \to \text{Hom}_R(L, N), \quad f_*(\varphi) = f \circ \varphi$$

is a homomorphism of abelian groups. This defines a functor

$$\text{Hom}_R(L, -): R	ext{-Mod} \to \text{Ab}$$

This functor is in general not exact. Take e.g. $R = \mathbb{Z}$, $L = \mathbb{Z}/2\mathbb{Z}$. We have a short exact sequence of abelian groups:

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

On the other hand the sequence

$$0 \to \text{Hom}_\mathbb{Z}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \text{Hom}_\mathbb{Z}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \text{Hom}_\mathbb{Z}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to 0$$

is not exact since $\text{Hom}_\mathbb{Z}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \cong 0$ and $\text{Hom}_\mathbb{Z}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.
47.7 Proposition. Let \( R \) be a ring and let \( L \) be an \( R \)-module. If
\[
0 \rightarrow N \xrightarrow{f} M \xrightarrow{g} K \rightarrow 0
\]
is a short exact sequence of \( R \)-modules then
\[
0 \rightarrow \text{Hom}_R(L, N) \xrightarrow{f_*} \text{Hom}_R(L, M) \xrightarrow{g_*} \text{Hom}_R(L, K)
\]
is an exact sequence of abelian groups.

Proof. Exercise. \( \square \)

47.8 Theorem. Let \( R \) be a ring with identity and let \( P \) be an \( R \)-module. The functor \( \text{Hom}_R(P, -) \) is exact iff \( P \) is a projective module.

Proof. By Proposition 47.7 is suffices to show that \( P \) is a projective module iff for every epimorphism of \( R \)-modules \( g: M \rightarrow K \) the map
\[
g_*: \text{Hom}_R(P, M) \rightarrow \text{Hom}_R(P, K)
\]
is an epimorphism. This follows directly from Theorem 43.9. \( \square \)

47.9 Definition. Let \( \mathcal{C}, \mathcal{D} \) be categories. A contravariant functor \( F: \mathcal{C} \rightarrow \mathcal{D} \) consists of
\begin{enumerate}
\item an assignment
\[
\text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D}), \quad c \mapsto F(c)
\]
\item for every \( c, c' \in \mathcal{C} \) a function
\[
\text{Hom}_\mathcal{C}(c, c') \rightarrow \text{Hom}_\mathcal{D}(F(c'), F(c)), \quad f \mapsto F(f)
\]
such that \( F(id_c) = id_{F(c)} \) and \( F(gf) = F(f)F(g) \).\end{enumerate}
47.10 Example. Let $R$ be a ring and let $L$ be an $R$-module. For any homomorphism of $R$-modules $f : M \to N$ we have a map
\[ f^* : \text{Hom}_R(N, L) \to \text{Hom}_R(M, L), \quad f^*(\varphi) = \varphi \circ f \]
Moreover, $f^*$ is a homomorphism of abelian groups.
This defines a contravariant functor
\[ \text{Hom}_R(-, L) : R\text{-Mod} \to \text{Ab} \]

47.11 Note. The functor $\text{Hom}_R(-, L)$ is in general not exact. Take e.g. $R = \mathbb{Z}$, $L = \mathbb{Z}$. We have a short exact sequence of abelian groups:
\[ 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \]
On the other hand the sequence
\[ 0 \to \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \to \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \to 0 \]
is not exact.

47.12 Proposition. Let $R$ be a ring and let $L$ be an $R$-module. If
\[ 0 \to N \xrightarrow{f} M \xrightarrow{g} K \to 0 \]
is a short exact sequence of $R$-modules then
\[ 0 \to \text{Hom}_R(K, L) \xrightarrow{g^*} \text{Hom}(M, L) \xrightarrow{f^*} \text{Hom}(M, L) \]
is an exact sequence of abelian groups.

Proof. Exercise. \qed
47.13 Theorem. Let $R$ be a ring with identity and let $J$ be an $R$-module. The functor $\text{Hom}_R(-, J)$ is exact iff $J$ is an injective module.

Proof. By Proposition 47.12 is suffices to show that $J$ is an injective module iff for every monomorphism of $R$-modules $f: N \to M$ the map

$$f^*: \text{Hom}_R(M, J) \longrightarrow \text{Hom}_R(N, J)$$

is an epimorphism. This follows directly from Proposition 46.1. \qed

47.14 Note. Let

$$M_* = (\ldots \longrightarrow M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \ldots)$$

be a chain complex of $R$-modules. For any $R$-module $L$ we have the induced chain complex of abelian groups

$$\text{Hom}_R(M_*, L) = (\ldots \longrightarrow \text{Hom}_R(M_{i-1}, L) \xrightarrow{d_i^*} \text{Hom}_R(M_i, L) \longrightarrow \ldots)$$

Homology groups of the complex $\text{Hom}_R(M_*, L)$ are called cohomology groups of $M_*$ with coefficients in $L$. We denote:

$$H^i(M_*, L) := H_i(\text{Hom}_R(M_*, L))$$

If $L$ is an injective module then by Theorem 47.13 the functor $\text{Hom}_R(-, L)$ is exact. By (47.5) in such case we have

$$H^i(M_*, L) \cong \text{Hom}_R(H_i(M_*), L)$$