NAME: SOLUTIONS

Math 141A- Final Exam - December 8, 2014

Instructions: The exam is worth 150 points. You should not use any aids, electronic or paper, other than a writing utensil.

1. **(15 points)** Evaluate the following integrals:

a. $\int_0^\pi \sin(3t)dt.$ $= \frac{-\cos(3t)}{3}\Big|_0^\pi = -\frac{1}{3}(\cos(3\pi) - \cos 0) = -\frac{1}{3}(-1-1)$

$= 2/3$

b. $\int \frac{4x+2}{x^2+x-1}dx.$ $u = x^2+x-1 \quad du = 2x+1$

$= \int \frac{2}{u}du = 2\ln|u| + C = 2\ln|x^2+x-1| + C$

c. $\int_0^2 y^2\sqrt{1+y^3}dy$ $u = 1+y^3 \quad du = 3y^2dy \quad y=0 \to u=1 \quad y=2 \to u=9$

$= \int_0^9 \frac{1}{3}u^{1/2}du = \frac{2}{9}u^{3/2}\Big|_0^9 = \frac{2}{9}\cdot 27 = 54/9 = 6$

d. $\int x^3 + 2x^2 + \frac{1}{x}dx.$ $= \boxed{\frac{1}{4}x^4 + \frac{2}{3}x^3 + \ln|x| + C}$

e. $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$ $u = \ln x \quad du = 1/x\, dx \quad x=e \to u=1 \quad x=e^4 \to u=4$

$= \int_1^4 \frac{1}{\sqrt{u}}du = 2\sqrt{u}\Big|_1^4 = 2\cdot 2 - 2\cdot 1 = 2$

2. **(15 points)** a. Estimate the area under the curve $y = 1 + x^2$ for $0 \leq x \leq 2$ using 4 intervals and right hand endpoints (i.e. compute R_4.) Is your answer an overestimate or underestimate? Explain.

$$R_4 = \frac{1}{2}\left(f(1/2) + f(1) + f(3/2) + f(2)\right) = \frac{1}{2}\left(\frac{5}{4} + 2 + \frac{9}{4} + 5\right) = \frac{1}{2}\left(\frac{42}{4}\right) = 42/8$$

$$= 21/4$$

overestimate because $1+x^2$ is increasing on $[0,2]$

b. Now find the exact area by computing the following definite integral **using the definition as a limit of Riemann sums!**:

$$\int_0^2 1 + x^2 dx.$$

$\Delta x = \frac{2}{n}$

$x_i = \frac{2i}{n}$

You may find one of the following formulas useful:
$\sum_{i=1}^n i = \frac{n(n+1)}{2}$, $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \left(\frac{2i}{n}\right)^2\right)\frac{2}{n}$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{2}{n} + \frac{8i^2}{n^3}\right) = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} 1 + \lim_{n\to\infty} \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{2}{n}\cdot n + \lim_{n\to\infty} \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= 2 + \lim_{n\to\infty} \frac{16n^3 + \dots}{6n^3} = 2 + 8/3 = 14/3$$

Check: $\int_0^2 1+x^2 dx = x + \frac{x^3}{3}\Big|_0^2 = 2 + 8/3 = 14/3$

3. **(10 points)** State both parts of the Fundamental Theorem of Calculus.

Let $f(x)$ be continuous on $[a,b]$

1. $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

2. Suppose $F'(x) = f(x)$

Then $\int_a^b f(x)dx = F(b) - F(a)$

4. **(10 points)** Use the Fundamental Theorem of Calculus to find the derivative $g'(x)$:

a. $g(x) = \int_3^x e^{t^2}dt$.

$g'(x) = e^{x^2}$

b. $g(x) = \int_0^{x^2}\cos(t)dt$.

$g'(x) = 2x \cdot \cos(x^2)$

5. **(10 points)** Consider the following limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2} \cdot \frac{1}{n}.$$

Recognize this limit as a definite integral and then evaluate it by evaluating the integral.

This is $\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}x \Big|_0^1$

$= \tan^{-1}1 - \tan^{-1}0$

$= \pi/4 - 0 = \pi/4$

6. **(10 points)** Let $f(t) = t\sqrt{4-t^2}$. Find the absolute maximum and absolute minimum of $f(t)$ on the interval $[-1, 2]$.

$f' = \sqrt{4-t^2} + t \cdot \frac{-2t}{2\sqrt{4-t^2}} = \sqrt{4-t^2} - \frac{t^2}{\sqrt{4-t^2}}$

Set $f'=0 \Rightarrow \sqrt{4-t^2} = \frac{t^2}{\sqrt{4-t^2}} \Rightarrow 4-t^2 = t^2$

$\cancel{t^2+t=4}$ $t^2 = 4$

$t = \pm\sqrt{2}$

crit points in interval ~~@~~ $t = \sqrt{3}$

t	$f(t)$
-1	$\sqrt{3}$
$\sqrt{2}$	2
2	0

absolute max value $= 2$
absolute min value $= 0$

7. **(5 points)** Using the Fundamental Theorem of Calculus we conclude:

$$\int_{-1}^{2} 3x^{-4}dx = -x^{-3} \Big|_{-1}^{2} = -\frac{1}{x^3} \Big|_{-1}^{2} = -(1/8+1) = -9/8.$$

Why is this mistaken?

$\frac{3}{x^4}$ is not continuous on $[-1,2]$

so FTOC does not apply,

8. **(5 points)** If $f(x)$ represents the slope of a trail at a distance x miles from the start, what does $\int_2^6 f(x)dx$ represent?

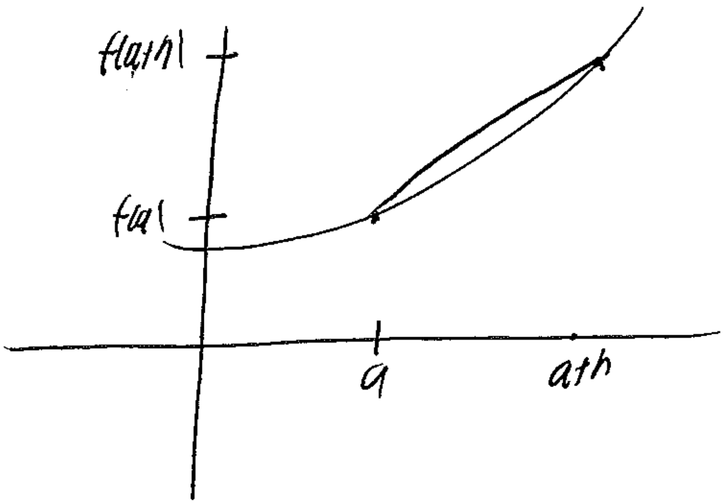
The net change in elevation between

mile marker 2 and 6

9. **(10 points)** The formula for the derivative of a function $f(x)$ at the value $x = a$ is given by:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if it exists. Explain in words, and using diagrams, where this formula comes from.



$\frac{f(a+h) - f(a)}{h}$ is the slope of the secant line connecting $(a, f(a))$ and $(a+h, f(a+h))$

As h gets smaller this slope approaches the slope of the tangent line.

Equivalently $\frac{f(a+h) - f(a)}{h}$ is an average rate of change of $f(x)$ and letting $h \to 0$ we get an instantaneous rate of change.

10. **(10 points)** Let $f(x) = x^3 + 2x + 1$.

a. Find the equation of the tangent line at $x = 2$.

$f' = 3x^2 + 2$

$f'(2) = 14 \leftarrow$ slope

point $(2, 13)$

$\boxed{y - 13 = 14(x - 2)}$

b. Prove $f(x)$ has one real root.

$f(-1) = -2$, $f(0) = 1$ So, since f is continuous on $[-1, 0]$, the Int Value Theorem guarantees a root in $(-1, 0)$

But $f'(x) = 3x^2 + 2$ is always > 0. If f had 2 roots Rolles' Thm would imply $f' = 0$ at some point between them

11. **(10 points)** Complete the definitions:

a. $\lim_{x \to a} f(x) = L$ if ...

For any $\varepsilon > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$

b. A function $f(x)$ is continuous at $x = a$ if ...

$$\lim_{x \to a} f(x) = f(a)$$

12. **(10 points)** Find $\frac{dy}{dx}$.

a. $y = xe^x$

$e^x + xe^x$

b. $y = 6x^2 + 3x + 5$

$12x + 3$

c. $y = (1 + \cos(x))^{10}$

$10(1+\cos x)^9(-\sin x)$

d. $y = \frac{1+x}{1-\ln(x)}$

$$\frac{1-\ln x - (1+x)(-\frac{1}{x})}{(1-\ln x)^2}$$

e. $xy^2 + \cos(y) = 2x$

$y^2 + 2xyy' - \sin y \, y' = 2$

$y'(2xy - \sin y) = 2 - y^2$

$$y' = \frac{2-y^2}{2xy - \sin y}$$

13. **(15 points)**

a. Find the equation of the line passing through $(-2, 1)$ and parallel to the line $2x+3y=2$.

slope $= -2/3$

$y - 1 = -\frac{2}{3}(x+2)$

b. Solve the inequality $x^2 - x - 6 \geq 0$.

$(x-3)(x+2)=0$

$+ \quad - \quad +$

$-2 \quad 3$

$(-\infty,-2] \cup [3,\infty)$

c. Find $\log_3(27)$.

3

d. Write $\frac{\cos x}{1+x} + \frac{x}{2+x}$ as a single fraction.

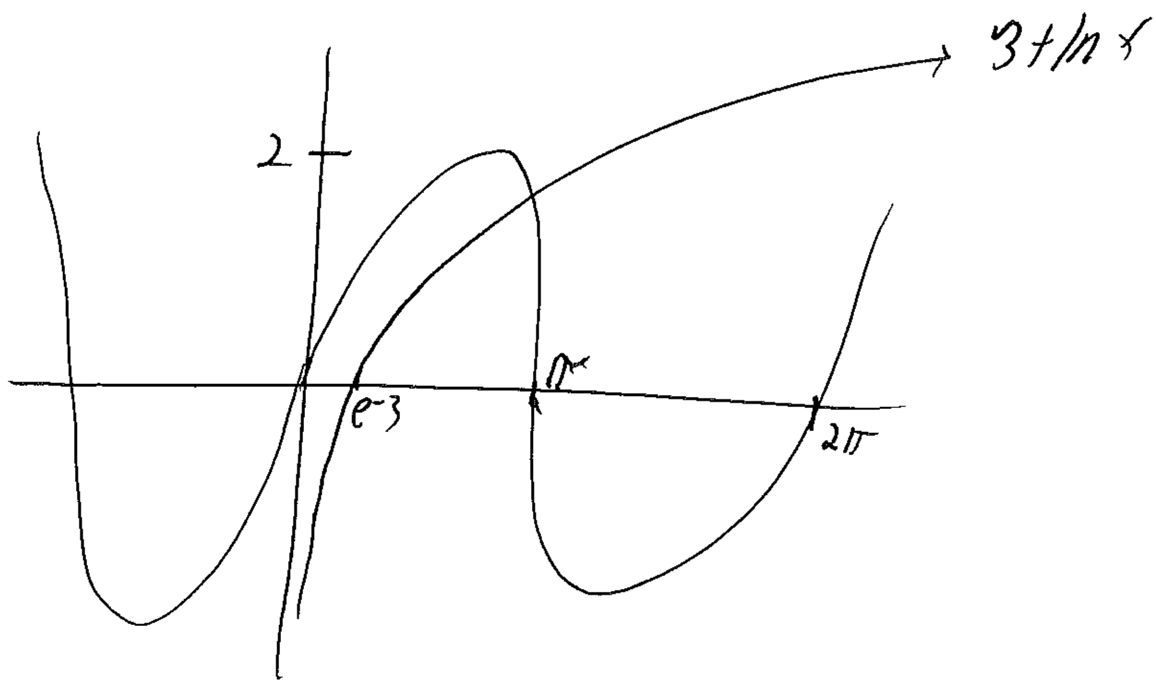
$$\frac{\cos x\,(2+x) + x(1+x)}{(1+x)(2+x)}$$

$3+\ln x = 0$

$\ln x = -3$

$x = e^{-3}$

e. Neatly sketch and label the graph of $y = 2\sin x$ and $y = 3 + \ln x$ on the same axes.



14. **(5 points)** Let $f(x) = 2x^2 + 3x + 1$. Find an x value c such that c satisfies the conclusion of the Mean Value Theorem for $f(x)$ on the interval $[1, 4]$.

$$\frac{f(4)-f(1)}{4-1} = \frac{45}{3}$$

$f' = 4x+3$

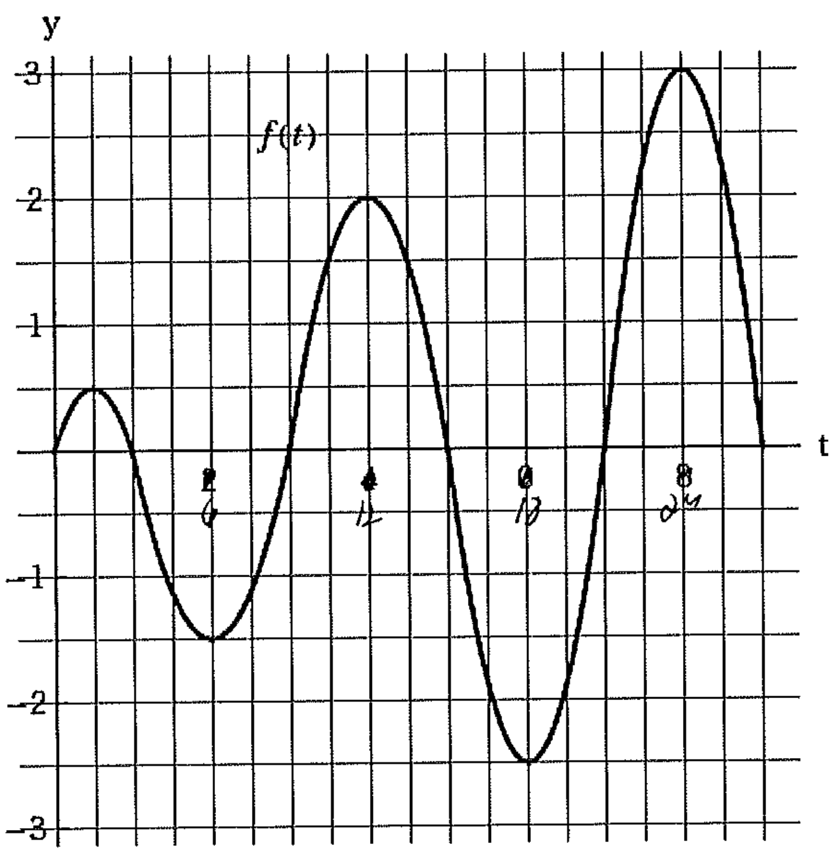
Set $4x+3 = 45/3$

$4x = 36/3 = 12$

$x = 3$

$c = 3$

15. **(10 points)** Below is the graph of $y = f(t)$ for $0 \le t \le 27$. (Please note the scale on the axis, each unit is 1.5)



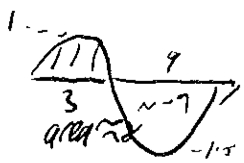
a. For what intervals is $f(t)$ increasing or decreasing? Concave up or concave down?

inc $(0,1.5) \cup (6,12) \cup (18,24)$

Dec $(1.5,6) \cup (12,18) \cup (24,27)$

b. Estimate $f'(15)$. -3

c. Estimate $\int_0^6 f(t)dt$. ≈ -6



d. Now define $g(x) = \int_0^x f(t)dt$. For what intervals is $g(x)$ increasing or decreasing?

increasing $(0,3) \cup (9,15) \cup (21,27)$

Decreasing $(3,9) \cup (15,21)$

e. What is $g'(12)$? $g'(12) = f(12) = 2$

f. At what values does $g(x)$ obtain local maximums? Global maximum?

local max $x = 3, 15, 27$

Global max $x = 27$