Lecture 20

Recall $y(t) = y(0)e^{kt}$ for exponential growth ($k>0$) or decay ($k<0$)

Ex Radioactive decay, mass $m(t) = m(0)e^{kt}$.

$$\tfrac{1}{2}\text{ life} = \frac{-\ln 2}{k}$$

Problem ½ life of cesium-137 is 30 years. Suppose 100 mg sample

- Find $m(t)$
- How long until 1 mg remains

A: $30 = \frac{-\ln 2}{k} \rightarrow k \approx -.02310$

$$m(t) = 100e^{-.02310t}$$

$1 = 100e^{-.02310t}$ $\qquad .01 = e^{-.02310t}$

$\ln(.01) = -.0231t$ $\qquad \boxed{t = 199.35 \text{ years}}$

Continuous interest

r = annual rate. Compound n times/year. t years

$$A = A_0\left(1+\frac{r}{n}\right)^{nt}.$$ Let $n\to\infty$ for continuous

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$= \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{n/r \cdot rt} \qquad \text{Let } m = \frac{n}{r}$$

$$= \left(\lim_{m\to\infty} A_0\left(1+\frac{1}{m}\right)^{m}\right)^{rt}$$

$$= A_0 e^{rt}$$

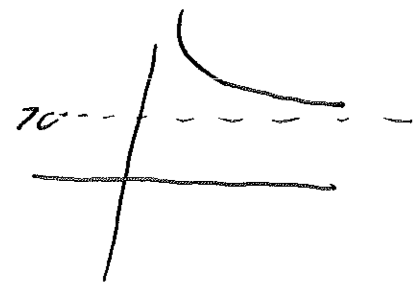
Conclude Continuous compounding, rate r, t years:

$$A(t) = A_0 e^{rt}.$$

Read on Own Newton's law of cooling: $\frac{dT}{dt} = k(T - T_s)$

Let $y = T(t) - T_s$ then $\frac{dy}{dt} = kt$

Ex 70° room, 150° coffee



Related Rates

Idea • Multiple quantities all depending on t.
- quantities are related by an equation
- Applying $\frac{d}{dt}$ implicitly gives relation between rates

Ex Radius of a sphere increases 2"/sec.
How fast is volume increasing when $r = 10$"?

Given $\frac{dr}{dt} = 2$"/sec Want $\frac{dV}{dt}$ when $r = 10$".

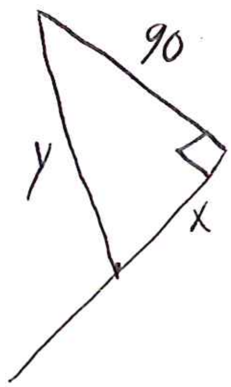
$V = \frac{4}{3}\pi r^3$. Apply $\frac{d}{dt}$:

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi \cdot 10^2 \cdot 2 = 800\pi \text{ in}^3/\text{sec}$$

A baseball diamond is square 90' / side

Batter runs toward first at 24 ft/sec.
At what rate is distance from 2^{nd} decreasing when halfway to 1^{st}



Given $\frac{dx}{dt} = -24$ ft/sec

Find $\frac{dy}{dt}$ when $x = 45$

$$x^2 + 90^2 = y^2$$

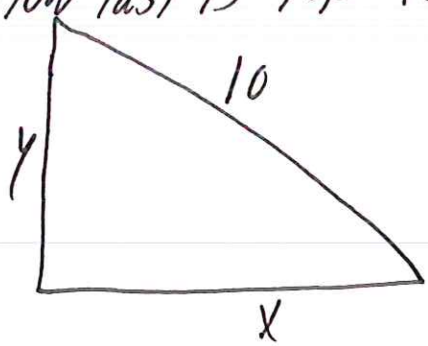
$$2x \frac{dx}{dt} = 2y \frac{dy}{dt}$$

$x = 45$
$y = 100.62$

$$90(-24) = 201.24 \left(\frac{dy}{dt}\right)$$

$$\frac{dy}{dt} = -10.7 \text{ ft/sec}$$

Ex Ladder 10 ft, pulled away from wall at 2 ft/sec
How fast is top falling when it is 3 ft high



$\frac{dx}{dt} = 2$ find $\frac{dy}{dt}$ when $y = 3$

$$x^2 + y^2 = 100$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$4x + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-4x}{2y}$$

what happens as $y \to 0$?

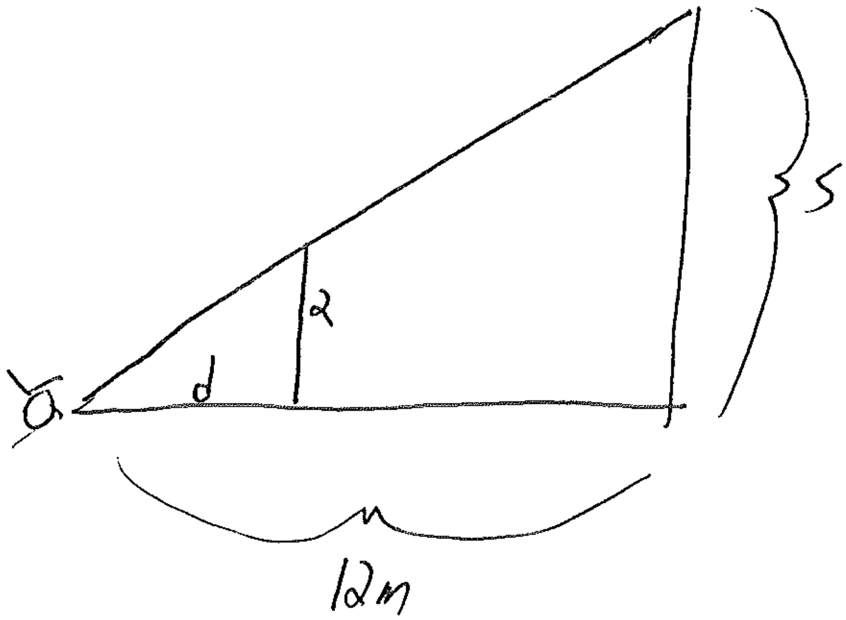
$y = 3$
$x = \sqrt{91}$

$$\frac{dy}{dt} = \frac{-4\sqrt{91}}{6} = -6.35$$

EX

Spotlight on ground! Man 2m tall walks 1.6 m/s
Shines on wall 12 m away. toward buildin.

How fast is length of his shadow decreasing when 4 m from buildin.



Given $\frac{dd}{dt} = 1.6\,m/s$ Find $\frac{ds}{dt}$ when $d = 8$.

Similar triangles!

$$\frac{2}{d} = \frac{s}{12} \qquad 24 = sd$$

$$0 = s\frac{dd}{dt} + \frac{ds}{dt}d$$

$$d = 8 \Rightarrow s = 3$$

$$0 = 3 \cdot 1.6 + \frac{ds}{dt} \cdot 8$$

$$\boxed{\frac{ds}{dt} = -4.8/8 \ m/s}$$