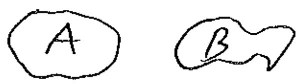
Lecture 32

Areas

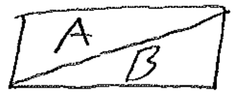
__Question__ What is area? Answer/Defin: Area of rectangle = $\ell \times w$.

__Property__ · If a set is a union of 2 disjoint pieces, area should add



Area$(A \cup B)$ = area A + area B

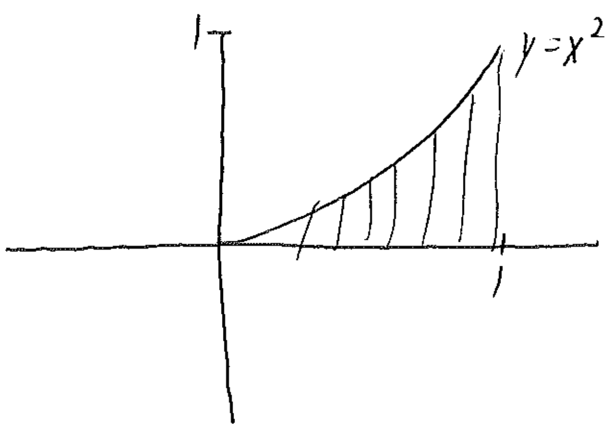
· What if they have 1-dimensional intersection?



→ Area triangle = $\frac{1}{2}bh$

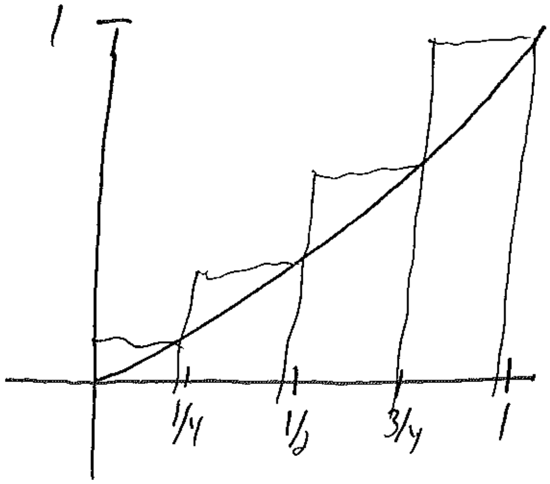
What if boundary is a fractal!

__Problem__ Find area under $y = x^2$ and above $[0,1]$



__Idea__ Similar to how we got slope of tangent line by approximation w/ secant lines

Approximate w/ rectangles, take a limit.



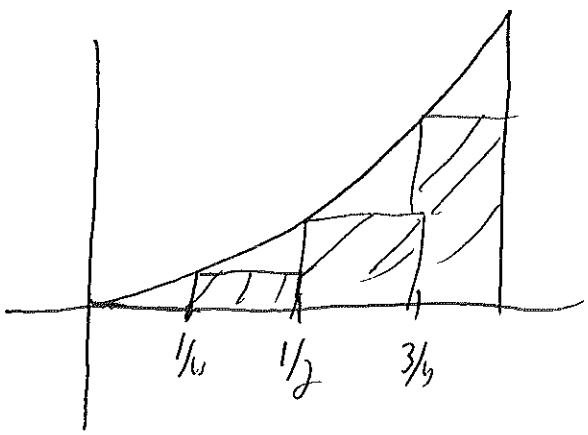
$f(x)=x^2$

- divide $[0,1]$ into intervals
- each has base $1/4$
- heights $f(1/4)=1/16$
  $f(1/2)=1/4$
  $f(3/4)=9/16$
  $f(1)=1$

$$\text{Area} \approx \frac{1}{4}\left(f(1/4)+f(1/2)+f(\tfrac{3}{4})+f(1)\right)$$
$$= \frac{1}{4}(1/16+1/4+9/16+1) = \frac{1}{4}\left(\frac{30}{16}\right) = \frac{30}{64} \boxed{=15/32}$$

$15/32$ is an overestimate

Could use left endpoints



$$A \approx \frac{1}{4}\left(f(0)+f(1/4)+f(1/2)+f(\tfrac{3}{4})\right)$$
$$= \frac{1}{4}(0+1/16+1/4+9/16)$$
$$L_4 = \frac{1}{4}\left(\frac{14}{16}\right) = 7/32$$

How to improve? More rectangles!

$$R_8 = \frac{1}{8}\left(f(1/8)+f(2/8)+\cdots+f(8/8)\right) \approx$$

$$R_n = \frac{1}{n}\left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \dots + \left(\frac{n}{n}\right)^2\right)$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \dots + n^2\right)$$

<u>Facts</u> $\displaystyle\sum_{i=1}^{n} i = 1+2+\dots+n = \frac{(n+1)n}{2} = \frac{n^2+n}{3}$

$\displaystyle\sum_{i=1}^{n} i^2 = 1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

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$$\lim_{n\to\infty} R_n = \frac{2}{6} = 1/3$$

<u>Other Options</u>

- midpoint
- lower & upper sums

<u>More generally</u>

$$\text{Area} = \lim_{n\to\infty}\left[f(x_1)\Delta x + \dots + f(x_n)\Delta x\right)$$

$x_i \in i^{th}$ int.