Lecture 5

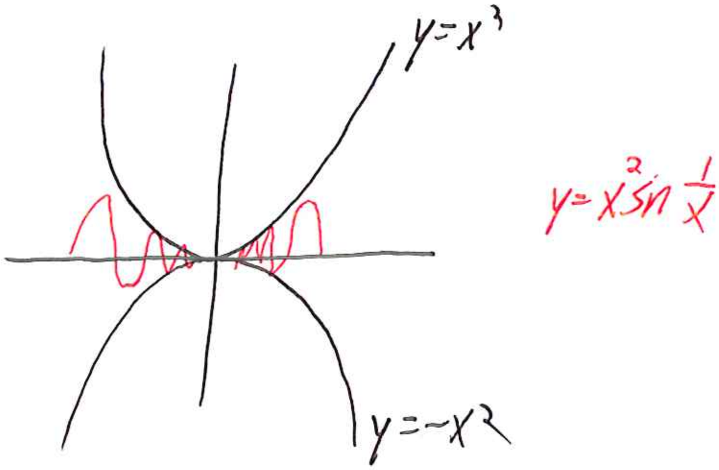
Review Limit laws

- many functions can sub in $x=a$ in domain.
- 1-sided versions, proofs in App

Ex $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$ since $\lim_{x\to 0} \sin(1/x)$ D.N.E. limit laws do not apply.

Need another tool.

Idea



Squeeze Theorem

Suppose $f(x) \le g(x) \le h(x)$ for x in a neighborhood of a (except possibly at a)

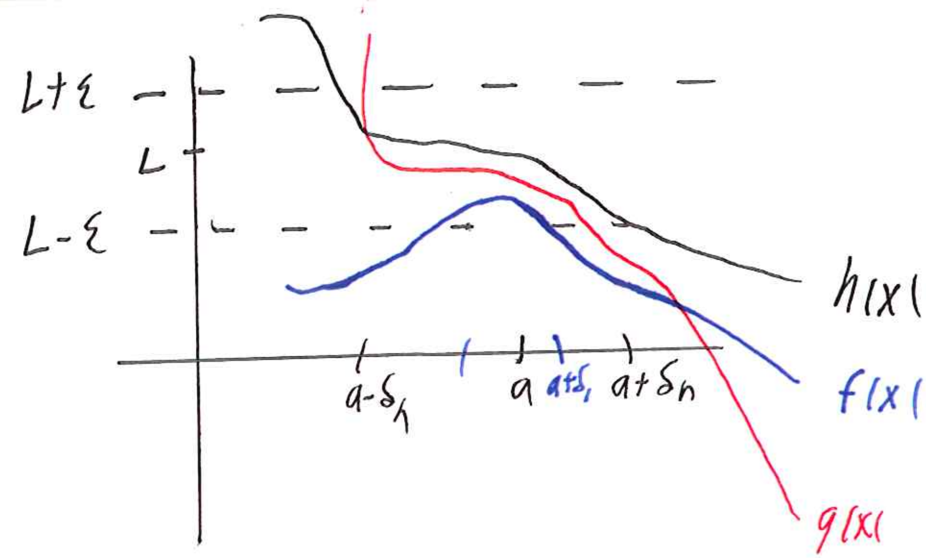
Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$.

Then $\lim_{x\to a} g(x) = L$

Proof Suppose $\varepsilon > 0$ is given. Choose δ_f so $0<|x-a|<\delta_f \Rightarrow |f(x)-L|<\varepsilon$

Choose δ_h so $0<|x-a|<\delta_g \Rightarrow |h(x)-L|<\varepsilon$

Smaller of δ_f or δ_h will work for $g(x)$



Remarks

- Don't need $g(x)$ squeezed between f & h for all x, only in a neighborhood of a.
- Choice of squeezing functions is key

Ex $\lim_{x \to 0} x^2 \sin(1/x) = 0$

Proof $-1 \le \sin(1/x) \le 1$ for all $x \neq 0$

$-x^2 \le x^2 \sin(1/x) \le x^2$ for all $x \neq 0$

But $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$ //

Ex $\lim_{x \to 0^+} \sqrt{x}\, e^{\sin(\pi/x)}$

between $1/e$ and e

Later $\cos\theta \le \dfrac{\sin\theta}{\theta} \le 1$ for θ near 0.

Continuity

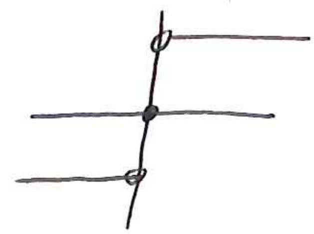
Informal/Imprecise: • $f(x)$ is continuous if can sketch graph w/out picking up pencil.

• Small changes in x produce "small" changes in $f(x)$.

Ex $f(x) = x^2$ continuous for all $-\infty < x < \infty$

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Not continuous at $x=0$



Def A function $f(x)$ is continuous at $x=a$ if $\lim_{x \to a} f(x) = f(a)$.

Rmk Need 3 things:
- $f(a)$ defined
- $\lim_{x \to a} f(x)$ exists
- they are equal

* continuous means you can evaluate limit by plugging in.

If not say $f(x)$ is discontinuous at $x=a$ or has a discontinuity at $x=a$

Examples

1. polynomials are continuous on $(-\infty, \infty)$

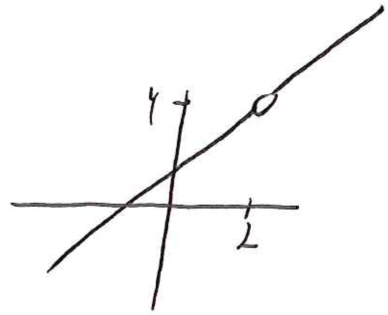
2. Rational Functions are continuous on their domains

   Ex  $f(x) = 1/x$ cont on $(-\infty, 0) \cup (0, \infty)$

3. e^x, $\sin x$, $\cos x$ continuous on $(-\infty, \infty)$
4. $\tan x$ etc. continuous on domain
5. Physical Quantities usually continuous

Types of discontinuities

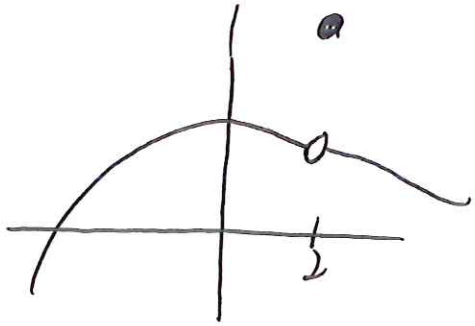
1. $f(x) = \frac{x^2-3}{x-2}$



$\lim_{x \to 2} f(x) = 4$

but $f(2)$ not defined

2.



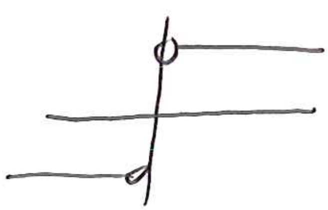
$\lim_{x \to 2} f(x)$ exists but $\neq f(2)$.

1 & 2 have <u>removable</u> discontinuities at $x=2$

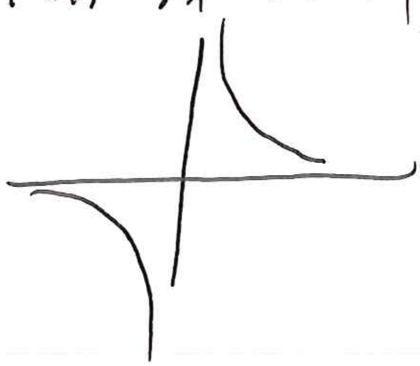
↳ limit exists but is not $= f(a)$!

3. $f(x) = \frac{|x|}{x}$ has a <u>jump discontinuity</u> at $x=0$

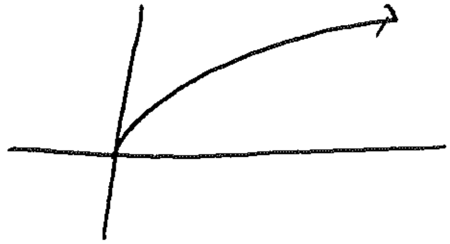
↳ limits from left & right exist but are $\neq$



4. $f(x) = 1/x$ has an infinite discontinuity at $x=0$.



Ex $y=\sqrt{x}$



officially not cont at $x=0$
since $\lim\limits_{x\to 0}\sqrt{x}$ DNE

<u>Def</u> $f(x)$ is <u>continuous from right</u> at $x=a$ if

$$\lim_{x\to a^+} f(x) = f(a)$$

" " " left " " :

$$\lim_{x\to a^-} f(x) = f(a)$$

<u>Def</u> Continuous on a closed interval!