Math 353 Homework #1- Due Wednesday 9/7/16

1. 2.2.1B There are $9 \cdot 10^3$ PINS. If no digit is allowed to be repeated there would be $9 \cdot 9 \cdot 8 \cdot 7$ PINS.

2. 2.2.4B There are 8! ways to put the red counters on the board. For each, we need to determine how many ways to put the green counters on so we have a legal arrangement. Given any such arrangement there is a unique permutation of the rows which will put the red counters on the diagonal, and the green counters will still be in a legal arrangement. Thus, WLOG, we can assume the red counters are on the diagonal, count the number of ways to put the green counters in, and then multiply by 8!.

Ignoring the red counters we have 8! ways to put the green counters in with one in each row and column. So we need to subtract off the number of arrangements where a green counter lies on the diagonal. We will do this by grouping such arrangements based on the number of counters on the diagonal. Notice if there are i greens on the diagonal then, removing those rows and columns, we have a solution to the $8-i \times 8-i$ problem. There are $\binom{8}{i}$ ways to pick which diagonal elements are overlapping. So let D_n be the number of ways to place green counters on an $n \times n$ board, with none on the diagonal. We conclude that:

$$D_n = n! - \binom{n}{1} D_{n-1} - \binom{n}{2} D_{n-2} - \binom{n}{3} D_{n-3} \cdots - \binom{n}{n-1} D_1 - \binom{n}{n} D_0 \tag{1}$$

where we define $D_0 = 1$, since the last term in (1) is the one arrangement where all the green counters are on the diagonal.

It's easy to calculate directly that $D_1 = 0$ and $D_2 = 1$. Now the recursion in (1) lets us determine every D_k once we know $D_1, D_2, \ldots, d_{k-1}$. We get:

$$D_3 = 2, D_4 = 9, D_5 = 44, D_6 = 265, D_7 = 1854, D_8 = 14833.$$

Thus the final answer is $8! \times 14833 = 598066560$.

3. 2.3.1B There are:

$$\binom{6}{4}\binom{8}{4}\binom{3}{1}\binom{4}{2}$$

ways to choose 4 out of six batsmen, then 4 out of 8 bowlers, 1 out of 3 wicketkeepers and 2 out of four all-rounders.

4. 2.3.2B We give a combinatorial proof. $C(2n, n)$ is the number of n element subsets of a $2n$ element set. Let's use $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_n\}$ as our set. To pick a subset of n elements we must first pick

a of the x_i, where $0 \leq a \leq n$. Then we must pick $n-a$ of the y_i. The number of ways to do this is $C(n,a)C(n,n-a)$. So dividing our subsets based on how many of the x_i we choose we get:

$$C(2n,n) = \sum_{r=0}^{n} C(n,r)C(n,n-r) = \sum_{r=0}^{n} C(n,r)^2$$

since $C(n,r) = C(n,n-r)$.

5. 2.3.3B Let X be a finite set and fix $a \in X$. Consider all possible subsets of X and define a function on these subsets as follows. Let $U \subseteq X$. If $a \in U$ then let $f(U) = U - \{a\}$. If $a \notin U$ then let $f(U) = U \bigcup \{a\}$. It is clear that $f(f(U)) = U$ so f is a bijection. It's also clear that if U has an odd number of elements then $f(U)$ has an even number (either one more or one less) and vice versa. So f gives a bijection between subsets with an even number and subsets with an odd number of elements. Thus we have proven that:

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots.$$

Collecting terms on the same side we get

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0.$$

6. 2.3.4B

Notice that $r(r-1)C(n,r)$ is the number of ways to pick an r element subset of an n element set and then from this subset pick an ordered pair of distinct elements, for example a president and a vice-president. There are $C(n,r)$ ways to pick the subset, r choices for the president and $r-1$ choices for the vicepresident. Thus $\sum_{r=0}^{n} r(r-1)C(n,r)$ is all possible committees with a president and a vice president.

Now lets count this set in a different way. First we pick the president and the vice president, there are $n(n-1)$ choices. Now we pick the rest of the committee, $r-2$ members. There are $C(n-2,r-2)$ ways to do this, where r runs from 2 to n. So we obtain:

$$\sum_{r=0}^{n} r(r-1)C(n,r) = \sum_{r=2}^{n} n(n-1)C(n-2,r-2).$$

Factoring out the $n(n-1)$ and reindexing the sum, this is equal to:

$$n(n-1)\sum_{r=0}^{n-2} C(n-2,r) = n(n-1)2^{n-2}.$$

So we conclude that:

$$\sum_{r=0}^{n} r(r-1)C(n,r) = n(n-1)2^{n-2}.$$

From part A we know

$$\sum_{r=0}^{n} rC(n,r) = n2^{n-1}.$$

Adding the previous two formulas gives the desired formula for $\sum_{r=0}^{n} r^2 C(n,r)$.

7. 2.4.2 A See back of book.

8. 2.4.2B There are $\binom{4n}{2n}$ total ways to draw $2n$ balls and of these there are $(\binom{2n}{n})^2$ ways to choose exactly n red and n blue. So the probability is:

$$\frac{(\binom{2n}{n})^2}{\binom{4n}{2n}}.$$