Math 353 Homework #5- Due Wednesday 10/5/16

1. 5.1.6B (Hint: Use Theorem 3.4 and proceed by induction on n). Let $X(n,k)$ be all possible products of $n-k$ integers from $\{1,2,\ldots,,k\}$, repeats allowed. Divide $X(n,k)$ into two subsets.

2. 5.2.3B (typo in the book here, should be $s(n,r)$ not $S(n,r)$.)

3. We now prove the Catalan numbers are uniquely determined by the recursion above. Suppose $\{d_n \mid n \geq 0\}$ is a sequence such that $d_0 = 1$ and:

$$d_n = \sum_{k=1}^{n} d_{k-1}d_{n-k}.$$

Prove by induction that $d_n = C_n$.

4. Suppose $w = w_1w_2\cdots w_n$ is a permutation of $\{1,2,\ldots,n\}$ written in one-line notation. Saw that w is *231-avoiding* if there do not exist indices $i < k < p$ such that $w_p < w_i < w_k$.

a. Write down the 231 avoiding permutations in S_4. (Hint: There are 14 of them)

b. Let S_n^{231} be the set of 231-avoiding permutations in S_n. Prove that the size of S_n^{231} satisfies the recursion and conclude from problem 4 that $\#(S_n^{231}) = C_n$. Hint: For an arbitrary 231-avoiding permutation w, consider the position of the letter n in w.

5. 5.3.2B. Hints below.

- First use the recursion to prove that $C_n > n+2$ for $n > 3$.
- Next prove from the definition that $(n+2)C_{n+1} = (4n+2)C_n$
- Suppose C_n is prime. Prove that C_n divides C_{n+1}.
- Prove that n must then be ≤ 4.

6. 5.3.8A