Math 353 Homework #5- Due Wednesday 10/5/16

1. **5.1.6B** Let's define $X(n,k)$ to be all the possible products of $n-k$ integers taken from $\{1,2,\dots,k\}$, repeats allowed. We must show that for $1 \le k < n$ that

$$\sum_{a \in X(n,k)} a = S(n,k).$$

We proceed by induction on n. It's easy to see that $X(2,1) = \{1\}$ and $S(2,1) = 1$ so the base case holds.

Now divide $X(n,k)$ into two subsets. The first, call it A, is the products that do not include the integer k. The second, call it B, are the products including k. Now A is all products of $n-k$ integers taken from $\{1,2,\dots,k-1\}$ so by induction the elements in A add up to $S(n-1,k-1)$. Each element of B is of the form $k*y$ where y is a product of $n-k-1$ integers taken from $\{1,2\dots,k\}$. So by induction the elements in B add up to $kS(n-1,k)$. Since $X(n,k) = A \sqcup B$ we have:

$$\sum_{a \in X(n,k)} a = S(n-1,k-1) + kS(n-1,k)$$

which equals $S(n,k)$ by Theorem 3.4.

2. **5.2.3B (typo in the book here, should be $s(n,r)$ not $S(n,r)$.)**
The Stirling number $s(n+1,k+1)$ is the absolute value of the coefficient of x^{k+1} in $[x]_{n+1}$. Cancelling the first x we see it is the coefficient of x^k in $(x-1)(x-2)\cdots(x-n)$. Now do the substitution $y = x-1$. Then:

$$(x-1)(x-2)\cdots(x-n) = y(y-1)(y-2)\cdots(y-n+1) = [y]_n.$$

Thus $s(n+1,k+1)$ is the coefficient of x^k in $[y]_n$. However by definition we have

$$[y]_n = \sum_{r=1}^{n}(-1)^{r-1}s(n,r)y^r = \sum_{r=1}^{n}(-1)^{r-1}s(n,r)(x-1)^r. \qquad (1)$$

Notice there is no x^k term in (**??**) until $r=k$ and the coefficient of x^k in $(x-1)^r$ is $(-1)^{r-k}\binom{r}{k}$, by the binomial theorem. So the coefficient of x^k in (**??**) is:

$$\sum_{r=k}^{n}(-1)^{r-1}s(n,r)(-1)^{r-k}\binom{r}{k} = (-1)^{-1-k}\sum_{r=k}^{n}s(n,r)\binom{r}{k}.$$

So taking absolute value we get:

$$s(n+1,k+1) = \sum_{r=k}^{n}\binom{r}{k}s(n,r)$$

as desired. **Note: I have not been able to find a combinatorial proof but there must be one!**

3. This is an easy induction proof, $d_0 = C_0 = 1$. Now suppose $d_i = C_i$ for $i < n$. Then:

$$d_n = \sum_{k=1}^{n} d_{k-1}d_{n-k} = \sum_{k-1}^{n} C_{k-1}C_{n-k}$$

by the inductive hypothesis. But the latter sum is just C_n since C_n satisfies the same recursion formula.

4.

a. In one-line notation the 231 avoiding permutations in S_4 are:

$$\{1234, 1324, 2134, 3124, 3214, 1243, 2143, 1423, 1432, 4123, 4132, 4213, 4312, 4321\}.$$

b. Let A_n be the number of 231-avoiding permutations in S_n. Let $1 \le k \le n$ and consider the set of 231-avoiding permutations in S_n with $\sigma(k) = n$. So in one line notation σ looks like:

$$???????n * * * * * **$$

where there are $k-1$ question marks and $n-k$ stars. The question marks must represent the numbers $1, 2, \ldots, k-1$ and the stars are $k, k+1, \ldots, n-1$, because as soon as a number $\le k-1$ appears to the right of n then a number $\ge k$ will be to the left and the forbidden 231 pattern appears.

Thus σ is 231 avoiding if the question marks form a 231 avoiding permutation of $\{1, 2, \ldots, k-1\}$ and the stars form a 231 avoiding permutation of $\{k, k+1, \ldots, n-1\}$. There are $A_{k-1}A_{n-k}$ choices (where A_0 is defined to be 1). This proves that:

$$A_n = \sum_{k=1}^{n} A_{k-1}A_{n-k}.$$

Since $A_0 = A_1 = 1$, the previous problem proves $A_n = C_n$.

5. 5.3.2B. Hints below.

- First use the recursion to prove that $C_n > n + 2$ for $n > 3$.
- Next prove from the definition that $(n + 2)C_{n+1} = (4n + 2)C_n$
- Suppose C_n is prime. Prove that C_n divides C_{n+1}.
- Prove that n must then be ≤ 4.

Proof: We have the recursion:

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0.$$

The RHS has n terms and once $n > 3$ we know two of them at least are > 1. This implies that for $n > 3$ that $C_n > n + 2$.

Now from the definition $C_n = \frac{\binom{2n}{n}}{n+1}$ it follows that:

$$\begin{aligned} C_{n+1} &= \frac{\binom{2n+2}{n+1}}{n+2} \\ &= \frac{\frac{(2n+2)!}{(n+1)!(n+1)!}}{n+2} \\ &= \frac{(2n+2)(2n+1)(2n)!}{(n+1)(n+1)n!n!(n+2)} \\ &= \frac{(4n+2)(2n!)}{(n+1)(n+2)n!n!} \\ &= \frac{4n+1}{n+2}C_n \end{aligned}$$

Now assume C_n is prime. From above we have that $(n+2)C_{n+1} = (4n+2)C_n$. Since we know $C_n > n+2$, then C_n must divide C_{n+1} since it cannot divide $n + 2$. So $C_{n+1} = kC_n$ for some k. This gives:

$$4n + 2 = k(n + 2)$$

which is impossible if $k \ge 4$. So $k = 1, 2, 3$ which forces n to be ≤ 4. Thus the only prime Catalan numbers are C_2 and C_3.

6. See back of the book