Math 353 Homework #6- Due Wednesday 10/8/14-SOLUTIONS

1. **Proof:** Let m be the number of partitions of n into exactly k parts ($m = p_k(n) = p_{k-1}(n)$). Suppose we have a solution of $x_1 + x_2 + \cdots + x_k = n$ with all the x_k positive. We know there are $C(n-1, k-1)$ solutions from Theorem 3.2. Given a solution, if we arrange the x_i in nonincreasing order, we get a partition of n with exactly k parts. For example $3+4+2+1 = 10$ becomes the partition $(4, 3, 2, 1)$. Now for each such partition there are *at most* $k!$ solutions of the equation that give the same partition, namely we reorder the $\{x_i\}$ in all possible ways. For example the solution $3+4+2+1 = 10$ has 4! different reorderings, all giving the same partition $(4, 3, 2, 1)$. However for the solution $1+3+3+3 = 10$ there are only 4 distinct reorderings giving the partition $(3, 3, 3, 1)$, namely

$$1+3+3+3 = 10, 3+1+3+3 = 10, 3+3+1+3 = 10, 3+3+3+1 = 10.$$

Thus $m \cdot k!$ overcounts the number of solutions to $x_1 + x_2 + \cdots + x_k = n$, and hence $m \cdot k! \geq C(n-1, k-1)$, so $m \geq \frac{1}{k!}C(n-1, k-1)$ as desired.

2. **Proof:** It seems from the hint that the authors did not mean to require the partitions have distinct parts. So let X be all partitions of n with smallest part k, so $\#X = s_k(n)$. Split X into two sets, A and B. Let A be those partitions with exactly one part equal to k, i.e. of the form $(\ldots, a, k)$ with $a > k$. Let B be those partitions with at least two parts of size k, so of the form $(\ldots, k, k)$. Adding one to the final part gives a bijection between A and partitions of $n+1$ with smallest part equal to $k+1$, so $\#A = s_{k+1}(n+1)$. For partitions in B we can remove the last part k and have remaining a partition of $n-k$ which still has smallest part equal to k. For any such partition, adding a part equal to k gives us an element of B, so $\#B = s_k(n-k)$. Since X is the disjoint union of A and B we have $s_k(n) = s_{k+1}(n+1) + s_k(n-k)$.

3. **Solution:** Here are the partitions of 20 using only 1,4,6,9,11,14,16,19. There are 31 of them:

$(19, 1), (16, 4), (16, 1^4), (14, 6), (14, 4, 1, 1), (14, 1^6), (11, 9), (11, 6, 1^3), (11, 4, 4, 1), (11, 4, 1^5), (11, 1^9),$ $(9, 9, 1, 1), (9, 6, 4, 1), (9, 6, 1^5), (9, 4, 4, 1^3), (9, 4, 1^7), (9, 1^{11}), (6, 6, 6, 1, 1), (6, 6, 4, 4), (6, 6, 4, 1^4), (6, 6, 1^8),$ $(6, 4^3, 1^2), (6, 4, 4, 1^6), (6, 4, 1^{10}), (6, 1^{14}), (4^5), (4^4, 1^4), (4^3, 1^8), (4^2, 1^{12}), (4, 1^{16}), (1^{20})$

Here are the partitions of 20 with parts differing by at least 2. Also 30. For use in the 2nd part of the problem note that 20 of them do not use 1.

$(20), (19, 1), (18, 2), (17, 3), (16, 4), (16, 3, 1), (15, 5), (15, 4, 1), (14, 6), (14, 5, 1), (14, 4, 2), (13, 7),$ $(13, 6, 1), (13, 5, 2), (12, 8), (12, 7, 1), (12, 6, 2), (12, 5, 3), (11, 9), (11, 8, 1), (11, 7, 2),$ $(11, 6, 3), (11, 5, 3, 1), (10, 8, 2), (10, 7, 3), (10, 6, 4), (10, 6, 3, 1), (9, 7, 4), (9, 7, 3, 1), (9, 6, 4, 1), (8, 6, 4, 2)$

Solutions Part II: Partitions of 20 using only 2,3,7,8,12,13,17,18 are below, there are 20, as expected:

$(18, 2), (17, 3), (13, 7), (13, 3, 2, 2), (12, 8), (12, 3, 3, 2), (12, 2^4), (8, 8, 2, 2), (8, 7, 3, 2), (8, 3, 3, 3, 3),$ $(8, 3, 3, 2, 2, 2), (8, 2^6), (7, 7, 2^3), (7, 7, 3, 3), (7, 3^3, 2^2), (7, 3, 2^5), (3^6, 2), (3^4, 2^4), (3^2, 2^7), (2^{10})$

4. We define a bijection f on the partitions in question so that $f(f(\lambda)) = \lambda$ and $f(\lambda) \neq \lambda$. This will group the set into pairs $\{\lambda, f(\lambda)\}$ so we must have an even number in total. Suppose the first part of λ is bigger than the second, so for example $\lambda = (32, 8, 4, 4, 2)$. For

$f(\lambda)$ just replace the first part with two parts each half the size, so $f(\lambda) = (16, 16, 8, 4, 4)$. If the first two parts of λ are equal, just combine them into one part. For example if $\lambda = (16, 16, 8, 8, 4, 2, 1)$ then $f(\lambda) = (32, 8, 8, 4, 2, 1)$. It should be obvious that $f(\lambda) \neq \lambda$ and $f(f(\lambda)) = \lambda$. Also f preserves the property of having all parts a power of 2. So the set is divided into pairs $\{\lambda, f(\lambda)\}$ as desired. Note this proof fails for $n = 1$, where the statement is not true.

5. We want a bijection between partitions with odd parts and partitions with distinct parts. Suppose $\lambda = (a_1, a_2, \ldots, a_t)$ has distinct parts. Write each $a_i = 2^{b_i} c_i$ where c_i is odd. Create a partition $f(\lambda)$ by replacing a_i with 2^{b_i} copies of c_i and putting them in order. For example:

$$\lambda = (24, 14, 12, 6, 1) = (2^3 * 3, 2 * 7, 2^2 * 3, 2^0 * 1$$

then we get $(3, 3, 3, 3, 3, 3, 3, 3, 7, 7, 3, 3, 3, 3, 1)$ and putting in order gives:

$$f(\lambda) = (7, 7, , 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 1).$$

How we want a map in the other direction. This uses that fact that each number has a base two expansion. So let μ have all odd parts and suppose the odd number a occurs t times. Write $t = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_a}$ with $0 \leq i_1 < i_2 < \cdots < i_a$. Replacing a^t with $(2^{i_1}a, 2^{i_2}a, 2^{i_3}a, \ldots)$. Here is an example:

$$\mu = (11^{13}, 7^5, 3, 1^2).$$

Now $13 = 1 + 4 + 8$ and $5 = 1 + 4$ and $1 = 1$ and $2 = 2$ so we get parts $\{11, 11 * 4, 11 * 8, 7, 7 * 4, 3, 2\}$ for a partition $(88, 44, 28, 11, 7, 3, 2)$. Check if you do this twice you get back to where you started.

6. Done in class.