Math 519- Exam #2 Take Home problems: Due Tuesday 11/20/2007

Instructions: You may use your book and your class notes, no other outside material of any sort. Do not consult with anyone other than the instructor about these problems. Try to write nice clear proofs! Do not assume the answers must be long and/or difficult!

1.

a. A commutative ring R with identity is called a *local ring* if it has a unique maximal ideal. Prove that if R is a local ring with unique maximal ideal M, then every element of $R - M$ is a unit. Prove, conversely, that if R is a commutative ring with identity in which the set of nonunits forms an ideal M, then R is a local ring with unique maximal ideal M.

b. Let R be the set or rational numbers whose denominators (in lowest terms) are odd. Check that this is a subring of the rational numbers. Determine its units. Then use part a. to prove R is a local ring whose unique maximal ideal is the principal ideal generated by 2.

2. Let I be an ideal in a commutative ring R with identity. Recall the definition of the *nilradical* $N(I)$ from p.268 #42. Define the *Jacobson Radical* of I as

Jac I is the intersection of all the maximal ideals of R that contain I.

Jac R is defined by convention to be R. (Notice that Jac 0 is the intersection of *all* maximal ideals.)

a. Prove that Jac I is an ideal of R which contains I.

b. Prove that $N(I) \subseteq$Jac I.

c. Let $n > 1$. Describe Jac $n\mathbb{Z}$ in terms of the prime factorization of n.

3. Let R be a commutative ring with identity. A proper ideal $I \subset R$ is called *primary* if whenever $ab \in I$ then either $a \in I$ or $b^n \in I$ for some $n \geq 1$. (Note that the symmetry between a and b in this definition implies that if $ab \in I$ with neither a nor b in I then some power of both a and b lies in I.)

a. Show that prime ideals are always primary.

b. Find the primary ideals in $\mathbb{Z}$.

c. Show that if I is a primary ideal than $N(I)$ is a prime ideal.