Math 519- Exam #2 Take Home problems: Due Tuesday 11/20/2007

Instructions: You may use your book and your class notes, no other outside material of any sort. Do not consult with anyone other than the instructor about these problems. Try to write nice clear proofs! Do not assume the answers must be long and/or difficult!

1.

a. A commutative ring R with identity is called a *local ring* if it has a unique maximal ideal. Prove that if R is a local ring with unique maximal ideal M, then every element of $R - M$ is a unit. Prove, conversely, that if R is a commutative ring with identity in which the set of nonunits forms an ideal M, then R is a local ring with unique maximal ideal M.

Suppose R is a local ring with unique maximal ideal M. Let $a \notin M$. Consider the ideal (a). If it is proper then it lies in some maximal ideal, hence in M, which is a contradiction. Thus $(a) = R$ so $1 = ra$ for some r, i.e. a is a unit. Conversely suppose the set of nonunits forms an ideal M. Any ideal containing a unit must be all of R, so no proper ideal can contain a unit. Thus every proper ideal lies in M, so M is the unique maximal ideal.

b. Let R be the set of rational numbers whose denominators (in lowest terms) are odd. Check that this is a subring of the rational numbers. Determine its units. Then use part a. to prove R is a local ring whose unique maximal ideal is the principal ideal generated by 2.

Suppose a/b, c/d are in R. Then $a/b + c/d = (ad + bc)/bd$ and bd is still odd. Reducing the fraction to lowest terms will not change that. Similarly $a/b * c/d = ac/bd \in R$. An element a/b is a unit if its inverse, b/a is also in R. Thus the units are those fractions where both a and b are odd. The set of nonunits are thus fractions of the form $2a/b$ with b odd, i.e. the principal ideal (2).It is easy to check this is an ideal. In particular note that $2a/b * c/d = 2ac/bd$ and the 2 will not cancel since both b and d are odd. Since the nonunits form an ideal, R is a local ring with maximal ideal (2).

2. Let I be an ideal in a commutative ring R with identity. Recall the definition of the *nilradical* $N(I)$ from p.268 #42. Define the *Jacobson Radical* of I as

Jac I **is the intersection of all the maximal ideals of R that contain I.**

Jac R is defined by convention to be R. (Notice that Jac 0 is the intersection of *all* maximal ideals.)

a. Prove that Jac I is an ideal of R which contains I.

Intersections of ideals are ideals so Jac I is an ideal. Since it is the intersection of ideals all of which contain I, then it contains I.

b. Prove that $N(I) \subseteq$Jac I.

Suppose $I \subseteq M$ and M is a maximal ideal. Suppose further that $N(I) \not\subseteq M$. Recall that $N(I) + M$ is an ideal containing both $N(I)$ and M, so must be all of R by maximality of M. Thus

$1 = a + m$ where $a \in N(I)$ and $m \in M$. Suppose $a^n \in I$. Then

$$1^n = (a+m)^n = a^n + m * (\cdots)$$

and both a^n and $m * (\cdots)$ are in M, so $1 \in M$ a contradiction.

c. Let $n > 1$. Describe Jac $n\mathbb{Z}$ in terms of the prime factorization of n.

Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Then Jac $n\mathbb{Z} = (p_1 p_2 \cdots p_k)$.

3. Let R be a commutative ring with identity. A proper ideal $I \subset R$ is called *primary* if whenever $ab \in I$ then either $a \in I$ or $b^n \in I$ for some $n \geq 1$. (Note that the symmetry between a and b in this definition implies that if $ab \in I$ with neither a nor b in I then some power of both a and b lies in I.)

a. Show that prime ideals are always primary.

This is immediate from the definition. If $ab \in I$ and $a \notin I$ then $b^1 \in I$.

b. Find the primary ideals in $\mathbb{Z}$.

These are (p^n) for a prime p.

c. Show that if I is a primary ideal than $N(I)$ is a prime ideal.

Let $ab \in N(I)$. We must show a or b is in $N(I)$. We know some power $(ab)^n = a^n b^n \in I$. If $a^n \in I$ then $a \in N(I)$ and we are done. If not, since I is primary than some power of b^n, say $b^{nm} \in I$. This implies $b \in N(I)$. Thus $N(I)$ is prime.