Math 519- Exam #1 Take Home problems: Due Wednesday 10/10/2007

Instructions: You may use your book and your class notes, no other outside material of any sort. Do not consult with anyone other than the instructor about these problems. Try to write nice clear proofs!

1. Let G be a finite group and let p be a prime which divides the order of G. Define X to be the set of all p-tuples of elements of G whose product is the identity, i.e.:

$$X = \{(g_1, g_2, \ldots, g_p) \mid g_i \in G,\ g_1 g_2 \cdots g_p = e.\}$$

a. Show that X has $|G|^{p-1}$ elements. In particular the cardinality of X is divisible by p.

b. Show that a cyclic permutation of an element of X is also an element of X, i.e. if $(g_1, g_2, g_3, \ldots, g_p) \in X$ then so is $(g_2, g_3, \ldots, g_p, g_1)$ and $(g_3, g_4, \ldots, g_p, g_1, g_2)$, etc...

c. Define the relation $\sim$ on X by setting $\alpha \sim \beta$ if β is a cyclic permutation of α. Prove that $\sim$ is an equivalence relation on X.

d. Prove that an equivalence class contains a single element if and only if it is of the form $(g, g, \ldots, g)$ with $g^p = 1$.

e. Prove that every equivalence class contains either 1 or p elements. (This requires p to be prime, be sure you make clear why!)

f. Since the equivalence classes partition the set X, conclude that:

$$|G|^{p-1} = k + pd$$

where k is the number of equivalence classes with one element and d is the number of equivalence classes with p elements.

g. Since $\{(e, e, e, \ldots, e)\}$ is an equivalence class of size 1, conclude from (f) that there must be a nonidentity element $g \in G$ with $g^p = e$. (show $p \mid k$).

h. Conclude Cauchy's Theorem: Let G be a finite group and p a prime dividing the order of G. Then G contains an element of order p. (Cauchy's original 1845 proof was 9 pages, hopefully you have improved on this!)

i. Give an example of a finite group G and an integer d which divides $|G$ such that G does not have an element of order d.

2. Consider the group of order 16 with the following presentation:

$$QD_{16} = \langle \sigma, \tau \mid \sigma^8 = \tau^2 = e, \sigma\tau = \tau\sigma^3. \rangle$$

(called the *quasidihedral or semidihedral* group of order 16).

a. Use the relations to explain why every element of QD_{16} can be put in the form $\tau^i \sigma^j$ with $0 \leq i \leq 1$ and $0 \leq j \leq 7$.

b. Construct the Cayley Table for QD_{16}. Please order your elements $\{e, \sigma, \sigma^2, \ldots, \sigma^7, \tau, \tau\sigma, \ldots, \tau\sigma^7\}$.

c. This group has 3 subgroups of order 8. Prove that $< \sigma > \cong Z_8$, $< \tau, \sigma^2 > \cong D_4$ and $< \sigma^2, \sigma\tau >$ is isomorphic to the Quaternion group whose Cayley table is given on page 90. In each case simply give an isomorphism, you don't have to prove it works.

d. What is the center of QD_{16}?