Name: SOLUTIONS

Math 419/519 Midterm Exam #2 -November 12, 2007

1. **(15 points)** Complete the following:

a. A subgroup $H \leq G$ is *normal* if ...

$ghg^{-1} \in H \ \forall g \in G, h \in H$

b. Let R be a commutative ring and I be a proper ideal. I is *prime* if ...

I is proper and $ab \in I \Rightarrow a \in I$ or $b \in I$

c. An integral domain is a *principal ideal domain* if ...

Every ideal is principal, i.e. of the form (a).

d. Let R be a ring with unity. $u \in R$ is a *unit* of ...

$\exists\, u^{-1} \in R$ such that $uu^{-1} = u^{-1}u = 1$.

e. Let R be a ring. A subset $I \subseteq R$ is an *ideal* if ...

$I \neq \emptyset$ and $\forall a, b \in I, \forall r \in R$ then $a-b, ra, ar \in I$.

2. **(16 points)**

a. Give an example of an integral domain which is not a field.

$\mathbb{Z}$

b. Give an example of a noncommutative ring with no identity.

$M_2(2\mathbb{Z})$

c. Let $\mathbb{Q}[x, y]$ be the polynomial ring on two variables with rational coefficients. Find a prime ideal which is not maximal.

$\{0\}$

d. Give an example of a maximal ideal in the ring $\mathbb{Z}$.

$2\mathbb{Z}$

3. **(10 points)** Find a zero divisor in $Z_5[i] = \{a + bi \mid a, b \in Z_5\}$.

$(2+i)(2-i) = 5 = 0$ so

$(2+i)$ is a zero divisor

(as is $2-i$)

4. **(15 points)** True or false:

F a. Every factor group of a nonabelian group is nonabelian.

$S_n/A_n \cong \mathbb{Z}_2$

T b. Every factor group of a cyclic group is cyclic.

T c. Every element in a ring has an additive inverse.

F d. As a ring, $\mathbb{Z}$ is isomorphic to $n\mathbb{Z}$ for any $n \geq 1$.

Ex $2\mathbb{Z} \ncong 3\mathbb{Z}$, see HW

F e. The direct product of two integral domains is again an integral domain.

never

F f. $3x^2 + 6x - 12$ is irreducible over $\mathbb{Z}$.

$= 3(x^2+2x-4)$

T g. Every ideal in a ring is also a subring of the ring.

T h. Every field of characteristic zero contains a subfield isomorphic to $\mathbb{Q}$.

T i. Let R be commutative ring with unity. Every maximal ideal in R is also prime.

F j. Let $f(x), g(x) \in \mathbb{Z}_{10}[x]$. Then $\deg(f(x)g(x)) = \deg f(x) + \deg g(x)$.

Ex $(2x^5+x)(5x+1)$

5. **(14 points)** Let H be a normal subgroup of G and let $n = [G : H]$. Prove that $a^n \in H$ for all $a \in G$.

The quotient group G/H has order n.

Any element raised to the order of the group power is identity

Thus $(aH)^n = H$

$\quad\quad\;\; \|$

$\quad\quad a^n H$ so $a^n \in H$.

6. **(10 points)** Let $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}$. Let $I = (\sqrt{2})$, the ideal generated by $\sqrt{2}$. How many elements are in the ring $\mathbb{Z}[\sqrt{2}]/I$? Explain.

any $\sqrt{2}(a+b\sqrt{2})$ is 0 in the quotient

$\quad\quad\quad \|$

$\quad a\sqrt{2}+2b$ In particular $2 \equiv 0$ and $\sqrt{2} \equiv 0$.

Thus $0+I,\ 1+I$ is all of $\mathbb{Z}[\sqrt{2}]/I$.

2 elements!

7. **(10 points)** Show that $\mathbb{Z}_3[x]/(x^2+x+1)$ is not a field.

$x^2+x+1=(x+2)^2$. Thus

$x+2+I \neq 0$ but $(x+2+I)^2 = 0+I$.

$x+2+I$ is a zerodivisor, the quotient is not even an integral domain!

8. **(10 points)** Let $\phi : R \to S$ be a ring homomorphism. Let $J \subseteq S$ be an ideal. Recall that:

$$\phi^{-1}(J) = \{r \in R \mid \phi(r) \in J\}.$$

Prove that $\phi^{-1}(J)$ is an ideal in R.

Let $r_1, r_2 \in \phi^{-1}(J)$ and $r \in R$

So $\phi(r_1), \phi(r_2) \in J$. Then

$\phi(r_1 - r_2) = \phi(r_1) - \phi(r_2) \in J$ since J is a subring

So $\boxed{r_1 - r_2 \in \phi^{-1}(J)!}$

So $\phi(r r_1) = \phi(r)\phi(r_1) \in J$ since J is an ideal!

So $\boxed{r r_1 \in \phi^{-1}(J)!}$

$\phi(r_1 r) = \phi(r_1)\phi(r) \in J$

So $\boxed{r_1 r \in \phi^{-1}(J)!}$

So $\phi^{-1}(J)$ is an ideal.