

2.  $U$  is a subspace if and only if it is a vector space in its own right under the same operations. Almost all the axioms, for instance the various distributive laws, are inherited from  $V$ . We just need to check that  $U$  is closed under the operations, i.e. under addition and scalar multiplication.

3. See back.

6. The set is not linearly independent. Setting a linear combination equal to zero gives the equations

$$2a + c = a + b + c = 2b + c = 0.$$

Choosing  $a = b = 2$  and  $c = 1$  gives a nontrivial solution, remembering that we are working in  $Z_5$ .

13. No, it is not closed under addition, adding two polynomials of degree 2 may result in a polynomial of degree 1 or 0.

14. No it is not, it isn't closed under addition. For example  $(3, 4, 5)$  and  $(5, 12, 13)$  are in it but  $(8, 16, 18)$  is not.  $W$  is known as a *cone* in  $V$ , because it is closed under scalar multiplication.

18. Proving this is a subspace is straightforward.  $P$  is a plane through the origin. A basis is  $(2, 1, 0)$  and  $(3, 0, 1)$ .

22.  $p^n$ .

24. Both are straightforward, just check they are closed under addition and scalar multiplication.

27. See back.

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1.  $\mathbb{Q}(\sqrt[3]{5}) = \{a + b\sqrt[3]{5} + c\sqrt[3]{25} \mid a, b, c \in \mathbb{Q}\}$  by Theorem 20.3.

2. Done in class.

7. See back.

15. Let  $f(x) = x^p - a$ . Let  $E$  be an extension field of  $F$  in which  $f(x)$  has a root  $\alpha$ . Thus  $\alpha^p = a$ . By the freshman binomial theorem,

$$x^p - a = x^p - \alpha^p = (x - \alpha)^p.$$

If  $\alpha \in F$  then  $f(x)$  splits over  $F$ . Suppose that  $\alpha \notin F$ , we must show  $f(x)$  is irreducible over  $F$ . Suppose  $f(x) = g(x)h(x)$  with  $g(x), h(x) \in F[x]$  of positive degree. Then over  $E$  we can write  $g(x) = (x - \alpha)^m$  for some  $1 \leq m < p$ . But:

$$(x - \alpha)^m = x^m - m\alpha x^{m-1} \dots$$

Notice that  $0 \neq m \in F$  so if  $m\alpha \in F$  then so is  $\alpha$ . But we are assuming it is not, so this is a contradiction. Thus  $f(x)$  is irreducible over  $F$ .

16.  $x^4 + x + 1 = (x + \beta)(x + \beta^2)(x + \beta^2 + 1)(x + \beta + 1)$ .

29. Since -1 must be the sum of the roots, the other root must be  $-1 - \beta = 4 - \beta$ .

30. This follows from Theorem 20.5 since the derivative is 1 and this has no positive degree factors in common with  $x^4 + x + 1$ .