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12. If H contains 18 and 30 is must contain 6 which $18 + 18 - 30$. (Indeed it is an easy exercise that if a and b are in a subgroup of $\mathbb{Z}$ then so is the gcd of a and b.) Thus 6 and 40 in H give us 2 in H, since $2 = 2 * 40 - 13 * 6$. Thus H contains all the evens, if it had any odd numbers it would be all of $\mathbb{Z}$. Since H is proper we conclude $H = 2\mathbb{Z}$.

15, 17. See back.

29. Note that $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Thus in $SL(2, \mathbb{R})$, the matrix A has infinite order. In $SL(2, Z_p)$ it has order p since $p = 0$ in Z_p.

31, 33. See back.

44. Let $H = \{A \in GL(2, \mathbb{R}) \mid \det A \text{ is a power of 2 }\}$. H is nonempty since it contains the identity. If A has determinant 2^a and B has determinant 2^b then AB^{-1} has determinant 2^{a-b}, and so is in H. Thus H is a subgroup.

51. See back.

52. Let G be a finite group with more than one element, so choose $g \neq e$. Then g has order $n > 1$. Suppose p is a prime dividing n. Consider the element $x = g^{n/p}$. Notice that $x^p = e$ but $x \neq e$ since that would contradict the order of g being n. However if $x^a = e$ for $a < p$ then a and p are relatively prime, so $1 = qa + rp$ for integers q and r. This would imply:

$$\begin{aligned} x &= x^1 \\ &= x^{qa+rp} \\ &= (x^a)^q (x^p)^r \\ &= e^q e^r \\ &= e \end{aligned}$$

which is a contraction. Thus x has prime order p.

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1. See back

8. a. 5, 5, 5, 5 b. 3, 3 c. 15, 15, 15, 15

9. See back.

14. If G is cyclic with exactly three subgroups, then its order must have only 3 divisors. Thus the order must be p^2 and the divisors are $1, p, p^2$. Notice if two distinct primes divide $|G|$, then $|G|$ will immediately have at least 4 divisors, namely $1, p, q, pq$..

20. Suppose G is an abelian group of order 35 and every element satisfies $x^{35} = e$. Then every element has order $1, 5, 7$ or 35, by Cor. 2 p.75. If G has an element of order 35 then clearly G is cyclic. By Corollary on p.80, the number of elements of order 7 is divisible by 6, and the number of elements of order 5 is divisible by 4. Since G has 34 nonidentity elements. But 34 is not a multiple of 4 or 6. This means G has elements both of order 5 and order 7. Let x have order 5 and y have order 7. Then xy has order 35 and G is cyclic. *This argument* does not work for 33 elements since $\phi(3) = 2$ and 32 is divisible by 2. Thus it is in theory possible that the group has the identity and 32 elements of order 2. In reality however and abelian group of order pq for distinct primes p, q will be cyclic.

24.$a * a^j = a^j * a = a^{j+1}$. Thus anything in $< a >$ commutes with a so $< a > \leq C_G(a)$.

31, 49, 51. See back.

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2.Let $H \leq G$ and define $N_G(H) = \{x \in H \mid xHx^{-1} = H\}$. Clearly $e \in N_G(H)$ so it is nonempty. Now suppose $x, y \in N_G(H)$. Then

$$\begin{aligned} xyH(xy)^{-1} &= xyHy^{-1}x^{-1} \\ &= xHx^{-1} \text{since } y \in N_G(H) \\ &= H \text{since } x \in N_G(H) \end{aligned}$$

Thus $xy^{-1} \in N_G(H)$ so it is a subgroup.

3. See back.

4. a. Center is $\{e, a^2\}$.
   b. $cl(a) = \{a, a^3\}$
   c. $cl(b) = \{b, ba^2\}$
   d. Cyclic subgroups $\{e\}, \{e, a, a^2, a^3\}, \{e, a^2\}, \{e, b, a^2, ba^2\}, \{e, ba^3, a^2, ba\}$

5.See back.

18. Let A be abelian and T be the set of elements of finite order. Thus $e \in T$ and since $|x| = |x^{-1}|$, then T is closed under taking inverses. Finally if x has order n and y has order m then, since A is abelian, $(xy)^{mn} = (x^n)^m(y^m)^n = e$ so xy has finite order (which is actually the lcm of m and n). Thus T is a subgroup. The same is not true for nonabelian groups, see p.69 #28 for an example.