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1. See back.

2. The order of a k-cycle is k.

3. See back.

4. a. The order of (12)(356) is 6.
b. The order of (1753)(264) is 12.

5. See back.

12. Suppose α is even. Then by definition

$$\alpha = t_1 t_2 \cdots t_{2k}$$

where each t_i is a transposition, in particular then $t_i^{-1} = t_i$. Thus

$$\alpha^{-1} = t_{2k} t_{2k-1} \cdots t_2 t_1$$

is also a product of an even number of transpositions, so is even. The proof for odd is essentially identical, replacing $2k$ with $2k + 1$.

18. $\alpha = (12345)(678) = (15)(14)(13)(12)(68)(67)$.
$\beta = (23847)(56) = (27)(24)(28)(23)(56)$.

6. (12345)(678) is an element of order 15 in A_8

8. The maximum order of an element in A_{10} is 21, for example (1234567)(8910) is even and has order 21. Notice that S_{10} has elements of order 30, for example (12)(345)(678910), but they are odd so not in A_{10}.

29. See back. Notice that only a single 9-cycle in S_9 can cube to be a product of 3 disjoint 3 cycles.

31. See back.

55. See back.

45. Let $\alpha = t_1 t_2 \cdots t_{2k}$ be an arbitrary element in A_n. If we can show that a product of a pair of transpositions $t_1 t_2$ can be expressed as a product of 3 cycles then we are done, since we can repeat this k times to express α as such. However $(ab)(cd) = (abc)(bcd)$ if a, b, c, d are all different. Also $(ab)(ac) = (acb)$. Of course $(ab)(ab) = e$ so any repeats can be eliminated.

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2. Let $\psi \in Aut(\mathbb{Z})$, so $\psi(a + b) = \psi(a) + \psi(b)$. It follows that for any $n \in \mathbb{Z}$ that $\psi(n) = n\psi(1)$. So for $a \in \mathbb{Z}$ define $\psi_a(x) = ax$. Notice that $\psi_a(x + y) = \psi_a(x) + \psi_a(y)$ so ψ_a

is a group homomorphism and is injective. However its image is all multiples of a, so ψ is only surjective if $a = 1$ or -1. Of course ψ_1 is the identity map and $\psi_{-1} \circ \psi_{-1} = \psi_1$. Thus $\psi \in Aut(\mathbb{Z}) \cong \mathbb{Z}_2$,

7. See back.

10. Let $\alpha(g) = g^{-1}$. This map is clearly a bijection so α is an isomorphism if and only if it is a homomorphism if and only if $\alpha(xy) = \alpha(x)\alpha(y)$. This holds if and only if $(xy)^{-1} = x^{-1}y^{-1}\forall x, y \in G$. However in any group $(xy)^{-1} = y^{-1}x^{-1}$. Thus α is a homomorphism if and only if $x^{-1}y^{-1} = y^{-1}x^{-1}$ for all $x, y \in G$, i.e. if and only if G is abelian.

24. Define $\phi : G \to H$ by $\psi(a + b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$. Clearly ϕ is $1-1$ and onto H. Let $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$. Then $x + y = (a + c) + (b + d)\sqrt{2}$ so

$$\phi(x + y) = \begin{pmatrix} a + c & 2(b + d) \\ b + d & a + c \end{pmatrix} = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} + \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} = \phi(x) + \phi(y)$$

so ϕ is an isomorphism of groups under addition.

Notice that $xy = (ac + 2bd) + (ad + bc)\sqrt{2}$ so

$$\phi(xy) = \begin{pmatrix} ac + 2bd & 2(ad + bc) \\ ad + bc & ac + 2bd \end{pmatrix}.$$

Now check that $\phi(xy)) = \phi(x)\phi(y)$ so the isomorphism preserves multiplication as well.

27. See hint in the back and proceed exactly as in 24.