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9. See back.

14. It is not a homomorphism because it is not well defined. For example in Z_{12}, $3 = 15$. However $\phi(3) = 9$ and $\phi(15) = 45$ but 9 and 45 are not congruent mod 10!

15. See back. Recall that all the elements in a single coset of the kernel map to the same element.

16. If there was a homomorphism from $Z_8 \oplus Z_2$ onto $Z_4 \oplus Z_4$ it would have to be an isomorphism, it is assumed onto but both groups have 16 elements so it would also be one-to-one. However these groups are clearly not isomorphic, the one on the left has elements of order 8 why the one on the right has maximum order of an element being 4.

21. By the 1st $\cong$ theorem $Z_{30}/ker\phi$ is isomorphic to the image, which has 5 elements. Thus the kernel has 6 elements. Z_{30} has a unique subgroup of order 6, namely $\{0, 5, 10, 15, 20, 25\}$.

30. This follow from theorem 9. The kernel has order 5. Given a normal subgroup H of $Z_6 \oplus Z_2$ or order a, then $\phi^{-1}(H)$ is a normal subgroup of G or order $5|H|$. However $Z_6 \oplus Z_2$ is abelian so every subgroup is normal. It has subgropus of order $1, 2, 3, 4, 6$ and 12 which correspond to normal subgroups of G of orders $5, 10, 15, 20, 30, 60$.

47. See back. It can't be finite since the 1st isomorphism theorem would guarantee every prime divides its order.

49. See back.

54. Define a map $\phi : G \to G/H \times G/K$ by $\phi(g) = (gH, gK)$. Check that this is a homomorphism. Since the identity in $G/H \times G/K$ is (H, K), we see that g is in the kernel of ϕ exactly when $gH = H$ and $gK = K$, i.e. when $g \in H \cap K$. But $H \cap K = \{e\}$. Thus the first isomorphism theorem says $G \cong \phi(G)$ and $\phi(G)$ is a subgroup of $G/H \times G/K$.

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8. It is $\mathbb{Z}/k\mathbb{Z}$.

9. See back. To get the center just pick a matrix $Z \in Z(H)$. Write down the equation for Z to commute with an arbitrary element of H and see that this forces $a = c = 0$.

26. Let $z = a+bi$ be a complex number. Recall that the norm $|z| = \sqrt{a^2+b^2}$ so T is just the set of z of norm 1. Clearly R^+ and T are subgroups under multiplication. Note that their intersection is just $1 = 1+0i$, and they are normal because C^* is abelian. Finally for any $z \in C$ we have:

$$z = |z|\frac{z}{|z|} \in R^+T$$

so $C^* = R^+T$. This proves C^* is the internal direct product of R^+ and T.

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2. The unity in this ring is 6.

6. a. In Z_6, $3^2 = 3$.
 b. In Z_6, $3 * 2 = 0$ but neither 3 nor 2 is 0.
 c. In Z_12, $3 * 4 = 3 * 8$ but $4 \neq 8$.
 None of our n's are prime. Indeed the ring Z_p is a field, these properties all do hold in a field.

20. The elements in $M_2(Z)$ with multiplicative inverses are those with determinant ± 1.

22. If a and b are units then $(ab)(b^{-1}a^{-1}) = 1$ so ab is also a unit. Also since $(a^{-1})^{-1} = a$, we see that a^{-1} is a unit. Multiplication is associative and 1 is a unit, so we have checked all the axioms to see $U(R)$ is a group under multiplication.

28. $4 = 2 * 5$ in Z_6 so $4 \mid 2$. Also $7 = 3 * 5$ in Z_8, and $12 = 9 * 3$ in Z_{15}.

50. Let R Boolean. First notice that $(x+x) = (x+x)^2 = x^2+x^2+x^2+x^2 = x+x+x+x$. Subtracting gives $x+x = 0$, i.e. $x = -x$ for any x in a Boolean Ring. Now let $x, y \in R$. Then:

$$\begin{aligned} a+b &= (a+b)^2 \\ &= a^2+ab+ba+b^2 \\ &= a+ab+ba+b \end{aligned}$$

Subtracting gives $ab = -ba$ but we know $-ba = ba$ so $ab = ba$.