Name: SOLUTIONS

Math 461/561 Final Exam - December 17, 2015

1. **(20 points)** Complete the following:

a. Let L be a complex Lie algebra. Define the *Killing form* on L.

$$K(x,y) = \text{trace}(\text{ad}\,x \circ \text{ad}\,y)$$

b. A Lie subalgebra H of a Lie algebra L is said to be a *Cartan subalgebra* if ...

1. H is abelian
2. Every $h \in H$ is semisimple
3. H is maximal subject to 1 & 2

c. Define a *base* for a root system R.

$\alpha_1, \alpha_2, \ldots, \alpha_\ell$ is a base for $R \subset E$ if

it is a basis of E and every root $\alpha \in R$

is of form $\alpha = \sum_{i=1}^{\ell} c_i \alpha_i$ with all $c_i \geq 0$

or all $c_i \leq 0$,

and $c_i \in \mathbb{Z}$

d. Let H be a Cartan subalgebra of a Lie algebra L. Describe/define the *root space decomposition* of L.

For $\alpha \in H^*$ let $L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x \ \forall h \in H\}$

and let $\Phi = \{\alpha \overset{\neq 0}{\in} H^* \mid L_\alpha \neq 0\}$. Then because H is

abelian, every elt s.s., and $C_L(H) = H$ we get

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

e. Let L be a Lie algebra. Define a Lie algebra module V.

V is a vector space and for each $x \in L$

we have a linear map $x : V \to x \cdot V$ so the

map $L \to gl(V)$ is a Lie alg hom.

i.e. $[x,y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v$

f. Define a *derivation* of a Lie algebra L.

$\delta : L \to L$ linear such that

$$\delta([x,y]) = [\delta x, y] + [x, \delta y]$$

2. **(20 points)** True or false. If false, give a counterexample or explanation.

True a. The roots in the A_n root system all have the same length.

False b. Let L be the two-dimensional nonabelian Lie algebra over $\mathbb{C}$. Then any finite-dimensional L-module is completely reducible.

The adjoint rep is not completely reducible.

$L = \langle x, y \mid [x,y] = x \rangle$

$\operatorname{ad} x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ad

False c. Up to isomorphism, the Lie algebra $F_4(\mathbb{C})$ has a unique irreducible module in each dimension.

The smallest is 7 dim. Next smallest is 14.

True d. The B_2 root system has eight roots.

False e. Let G be a group which is finitely generated and also has finite exponent. Then G must be finite.

Discussed in class

False f. The Lie algebra $sl(n, F)$ is simple for any field.

char 2 $\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle$ is an ideal

False g. Finite dimensional irreducible $\mathfrak{sl}(3, \mathbb{C})$ modules are self-dual.

Natural module $\mathbb{C}^3 \not\cong$ its dual

3. **(10 points)** Let L be a finite-dimensional Lie algebra.

a. Define the radical $\operatorname{rad} L$ and the center $Z(L)$.

b. L is called *reductive* if $\operatorname{rad} L = Z(L)$. Show that if L is reductive then L is a completely reducible as an L module via the adjoint representation. (Hint: Weyl's theorem)

a. rad L is the largest solvable ideal. $Z(L) = \{x \in L \mid [x,y] = 0 \ \forall y \in L\}$

b. The kernel of the adjoint rep is $Z(L)$ so L as an L-module is also an $L/Z(L)$ module.

But $Z(L) = \operatorname{rad} L$ so $L/Z(L)$ is semisimple.

By Weyl's Thm L is completetely reducible as an $L/Z(L)$ module, and thus as an L module.

4. **(10 points)** Prove that $\mathfrak{sl}(2, F)$ is nilpotent in characteristic 2.

in char 2 $\quad sl(2,F) = \langle e, f, h \mid [h,e]=0,\ [h,f]=0,\ [e,f]=h \rangle$

So $L' = \operatorname{span}\langle h \rangle$

But $h \in Z(L)$ so $[L, L'] = 0$

and $sl(2,F)$ is nilpotent.

5. **(20 points)**

a. State Lie's Theorem

b. Let L be a solvable subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. Prove that the derived subalgebra $[L, L]$ is nilpotent.

c. Give a counterexample to *b.* in characteristic 2.

a. V f.d. $/\mathbb{C}$, $L \subseteq gl(V)$ solvable. Then $\exists$ a basis of V so every element of L is upper triangular.

b Let L be solvable. Choose a basis so $L \subseteq \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$ by Lie's Thm

Check for A, B upper triangular that $[A, B]$ is strictly upper triangular:

$$L' = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}$$

Then $$[L', L'] \subseteq \left\{ \begin{pmatrix} 0 & 0 & * \\ & \ddots & 0 \\ & & 0 \end{pmatrix} \right\}$$

etc.. so $L'^{n} = 0$

and L' is nilpotent.

c. See #6.4 ! in book

6. **(20 points)** Recall the construction of a root system type G_2. The ambient Euclidean space is

$$E = \{v = \sum_{i=1}^{3} c_i \epsilon_i \mid \sum c_i = 0\}.$$

There are 12 roots given by:

$$R = \{\pm(\epsilon_i - \epsilon_j), i \neq j\} \cup \{\pm(2\epsilon_i - \epsilon_j - \epsilon_k), \{i, j, k\} = \{1, 2, 3\}\}.$$

A base is given by

$$\{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 + \epsilon_3 - 2\epsilon_1.\}$$

a. Recall the fundamental dominant weights are defined by λ_1, λ_2 where $\lambda_i(h_{\alpha_j}) = \delta_{ij}$. Determine λ_1 and λ_2.

b. Let s_{α_1} the the corresponding simple reflection. Let $x = 2\epsilon_1 - \epsilon_2 - \epsilon_3$. Compute $s_{\alpha_1}(x)$ and express it in terms of the simple roots.

c. Give the Cartan matrix for G_2 with simple roots in order α_1, α_2.

c. $\langle \alpha_1, \alpha_2 \rangle = \dfrac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = \dfrac{2(-3)}{6} = -1, \quad \langle \alpha_2, \alpha_1 \rangle = \dfrac{2(\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} = \dfrac{2(-3)}{2} = -3$

So $C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

b $C^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ so $\lambda_1 = 2\alpha_1 + 3\alpha_2, \quad \lambda_2 = \alpha_1 + 2\alpha_2$

c. $\langle x, \alpha_1 \rangle = \dfrac{2(x, \alpha_1)}{(\alpha_1, \alpha_1)} = \dfrac{2(2\epsilon_1 - \epsilon_2 - \epsilon_3, \epsilon_1 - \epsilon_2)}{2} = \dfrac{2(2+1)}{2} = 3$

Now $s_{\alpha_1}(x) = x - \langle x, \alpha_1 \rangle \alpha_1 = x - 3\alpha_1$

But $x = -\alpha_2$ so

$$\boxed{s_{\alpha_1}(x) = -3\alpha_1 - \alpha_2}$$