Recovery of high order accuracy in radial basis function approximation for discontinuous problems

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Abstract

Radial basis function(RBF) methods have been actively developed in the last decades. The advantages of RBF methods are that these methods are mesh-free and yield high order accuracy if the function is smooth enough. The RBF approximation for discontinuous problems, however, deteriorates its high order accuracy due to the Gibbs phenomenon. With the Gibbs phenomenon in the RBF approximation, the L_∞ error remains only $\mathcal{O}(1)$. The main purpose of this paper is to show that high order accuracy can be recovered from the RBF approximation contaminated with the Gibbs phenomenon if a proper reconstruction method is applied. In this work, the Gegenbauer reconstruction method is used to reconstruct the RBF approximation for the recovery of high order accuracy. Several numerical examples presented in this work indicate that the Gegenbauer polynomials are *Gibbs complementary* to the RBF approximations and hence high order convergence can be recovered from the RBF approximations for discontinuous problems. These results also indicate that the Gegenbauer reconstruction method which was originally developed for the polynomial approximation works for non-polynomial basis methods such as the RBF method. Numerical examples including the linear and nonlinear hyperbolic partial differential equations are presented.

Key words: Radial basis functions, Discontinuous problems, Gibbs phenomenon, Post-processing, Gegenbauer reconstruction method, Gibbs complimentary.

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1 Introduction

It has long been observed that in global series approximations of functions with a discontinuity, the partial sums converge slowly away from the discontinuity and feature oscillations which do not decay even as the number of terms increases. This phenomenon was first observed in relation to Fourier series expansions and called 'the Gibbs phenomenon'. This term has been extended to refer to the corresponding behavior seen in other global approximations. Over the last fifteen years, it has been shown that global approximations may contain the higher order information within them, and that this higher order information can be extracted using suitable post-processing, or *reconstruction* [9,10]. For example, if we start with a global approximation in a given basis, which suffers from the Gibbs phenomenon due to the presence of a discontinuity, we can project the solution *in the smooth region* using an alternative basis, and recover high order convergence. Conditions for an appropriate basis for such method for the Fourier and Legendre polynomials were given in [9], where it was called a *Gibbs complementary basis*. It has been determined that the Gegenbauer polynomial basis is Gibbs complementary to the Fourier and Legendre expansions, and in fact for Gegenbauer expansions as well.

It has further been shown that Gegenbauer post-processing can recover high order accuracy in spectral and pseudo-spectral simulations of time dependent partial differential equations with discontinuities. Previous studies have demonstrated the recovery of high order accuracy in high order approximations of hyperbolic partial differential equations (PDEs) using the Gegenbauer reconstruction method proposed by David Gottlieb and his co-workers [9,10]. In [15], the Fourier spectral approximation was considered for discontinuous solutions of PDEs including the linear advection equations and inviscid Burgers' equation. In the presence of a local jump discontinuity, the convergence of the Fourier spectral method is only linear near the discontinuity and the solution is highly oscillatory. In [15], it was shown that such highly oscillatory solutions can be reconstructed with the Gegenbauer method such that the exponential accuracy is recovered *up to* the local jump discontinuity and consequently the Gibbs phenomenon is removed. The main objective of this paper is to show that the Gegenbauer reconstruction method also recovers order of accuracy in another class of global approximations, the radial basis function approximations.

Radial basis function (RBF) methods have been actively investigated for various problems such as image reconstruction, machine learning and numerical solution of differential equations. Collocation methods based on RBFs have the potential of yielding high order convergence for smooth problems. The accuracy of RBF methods can be enhanced by adaptively changing the shape parameters [11] which play a crucial role in the approximation with the RBF method, or by adaptively increasing the number of grid points in critical regions [3]. Furthermore, RBF methods have the advantage of being *meshless*: unlike methods based on trigonometric functions or polynomials, RBF approximations work well on any distribution of points. Most RBF method research has been focused on the approximation of smooth problem. However, if the problem considered is discontinuous, the RBF method suffers from the Gibbs phenomenon, as do other global approximations such as spectral methods and pseudo-spectral methods [5,11]. In the approximation of discontinuous problems with the RBF method, the maximum-norm error decays very slowly away from the discontinuity as we increase the number of center points N, and does not decay at all at the discontinuity even as N is increased. Since many problems in physics or engineering applications contain discontinuous features, it would be advantageous to develop a method of dealing with discontinuities in RBF approximations.

There have been several studies on the Gibbs phenomenon and related problems in RBF approximations [5,11,13]. In [11], the Gibbs phenomenon was investigated for multi-quadric(MQ) RBF approximation of a discontinuous function, and the shape-parameter adaptive method has been proposed to remove the Gibbs phenomenon. The Runge phenomenon [13] and the Gibbs phenomenon [5] were investigated using Gaussian RBFs, and in [5] it was shown that for Gaussian RBFs, the vanishing shape functions yield zeroth order approximation and consequently reduce the Gibbs oscillations. In [3], a mesh-adaptive RBF method was developed, to allow resolution of nonsmooth area of the approximations. This adaptive method, however, is only effective where the function is not truly discontinuous but only rapidly varying; it cannot handle a true discontinuity.

In this paper, we show through numerical demonstrations that although the MQ-RBF approximations of functions with local jump discontinuities converge slowly, they retain within them high order information which can be extracted using Gegenbauer reconstruction methods. We will show that the RBF approximation for discontinuous problems can be also post-processed so that high order accuracy and convergence can be recovered. The theoretical justification of the successful results of Gegenbauer post-processing for Fourier or polynomial approximation can be found in [9], where it was shown that any polynomial approximation with local jump discontinuities can be reconstructed in the smooth regions with the reconstruction method based on the Gibbs complementary basis, if the approximating coefficients are bounded. However, the RBF functions such as MQ-RBFs are not polynomials and so the results of this paper are significant since they imply that the reconstruction of a *nonpolynomial* basis approximation can also recover high order accuracy.

Furthermore, we demonstrate that just as stable spectral and polynomial pseudospectral simulations of time-dependent partial differential equations with discontinuous solutions (such as hyperbolic conservation laws) retain within them high order information which can be recovered through Gegenbauer post-processing [15], so too can Gegenbauer post-processing recover accuracy in results from stable MQ-RBF simulations of such problems. This paper, in the tradition of [15] and [8] , demonstrates this fact numerically; a careful mathematical analysis and justification will be considered in our future work.

The paper is composed of the following sections. In Section 2, the Gegenbauer reconstruction method is briefly explained. In Section 3, the general collocation method for time dependent problems is explained. In Section 4, the RBF method is explained. In Section 5 the Gegenbauer reconstruction method for the RBF methods is introduced for general RBF basis functions. For this, the RBF-Fourier Gegenabeur and the direct RBF-Gegenbauer reconstruction methods are introduced. In Section 5, several numerical examples are presented including the linear and nonlinear hyperbolic problems. The numerical examples presented in this section clearly show that the Gegenbauer reconstruction method recovers the high order convergence of the RBF approximations for discontinuous problems.

2 Gegenbauer Reconstruction

We begin with a global approximation of a function $f(x)$, for example the Fourier partial sum

$$f_N(x) = \sum_{k=-N}^{N} \hat{f}_k \exp(ik\pi x)$$

which approximates the function $f(x)$ on the domain $x \in (-1, 1)$. We call this approximation *global* because the Fourier coefficients $\hat{f}_k$ are calculated using information about the function from the entire domain $(-1, 1)$. If $f(x)$ is periodic and smooth enough everywhere in the domain $(-1, 1)$, then the partial sum $f_N(x)$ converges exponentially to the function $f(x)$ as N increases, for any point $x \in (-1, 1)$. However, in the case where there is a discontinuity somewhere in the domain, say at a point $x_d \in (-1, 1)$, the Fourier coefficients (due to their global nature) are all affected by this discontinuity. In this case, the Fourier partial sum $f_N(x)$ converges only linearly to the function $f(x)$ away from the discontinuity, i.e.

$$|f(x) - f_N(x)| \sim \frac{1}{N}$$

for any point x which is far enough from x_d. Worse, the partial sum forms oscillations near the discontinuity which do not diminish as we increase N, and so the maximum norm error

$$\max_{x\in(-1,1)} |f(x) - f_N(x)|$$

does not decay at all.

However, each slowly converging partial sum $f_N(x)$ contains within it enough information to allow us to construct a new, rapidly converging, approximation f_N^m on a subinterval (a, b) over which $f(x)$ is smooth. The Gegenbauer reconstruction method has been developed by D. Gottlieb and his coworkers and shown to recover spectral accuracy of Fourier approximations of functions resulting from non-periodicity or a local jump discontinuity inside the domain [9,10]. To do this, we project the Fourier partial sum using the Gegenbauer basis in the interval of smoothness. First, we compute the first $(m + 1)$ Gegenbauer coefficients, using only the Fourier partial sum and the known Gegenbauer polynomials

$$\tilde{g}_k^\lambda = \frac{1}{h_k^\lambda} \int_{-1}^{1} \left(1 - \xi^2\right)^{\lambda - \frac{1}{2}} f_N(\xi) G_k^\lambda(\xi) d\xi, \qquad k = 0, ..., m$$

where G_k^λ are the Gegenbauer polynomials, and $\tilde{g}_k^\lambda$ are the corresponding expansion coefficients with their normalization factors h_k^λ, and the variable $\xi = \frac{2x-a-b}{b-a}$ takes the sub-interval of smoothness (a, b) into the domain $(-1, 1)$. We then use these coefficients to reconstruct the function:

$$f_N^m(x) = \sum_{k=0}^{m} \tilde{g}_k^\lambda G_k^\lambda(\xi(x)), \quad x \in (a, b).$$

The remarkable part is that after this simple process, the reconstructed approximation converges exponentially everywhere in the domain of smoothness, in the sense that

$$\sup_{a<x<b} |f(x) - f_N^m(x)| \sim C \exp(-qN), \quad q > 0,$$

for appropriate choices of λ and m where C is a constant independent of λ and m. To recover exponential convergence, we must choose $\lambda = \alpha N$ and $m = \beta N$, where α and β are positive constants and $\alpha < 1$ and $\beta < 1$. However, there is no general theory yet to help us determine the optimal choice of α and β , and this is a difficulty in the implementation of Gegenbauer reconstruction methods. Another difficulty arises in the need to define the interval of smoothness. Typically, the discontinuity location is unknown, but needs to be avoided for the reconstruction. In these cases, edge-detection methods have been identified which allow us to find the intervals of smoothness.

This theory applies not just to the Fourier basis, but also to polynomial expansions. Furthermore, It can be shown analytically that these results extend to stable spectral (i.e. Fourier basis) or pseudospectral (i.e. polynomial basis) simulations of linear time dependent PDEs with discontinuous solutions, such as the linear advection equation. In other words, even after an approximation to a discontinuous function has been evolved in time, it is possible to extract from it a high order solution using Gegenbauer post-processing. To prove this for nonlinear problems is difficult, however numerical studies have demonstrated the recovery of high order accuracy in numerical approximations of nonlinear hyperbolic PDEs using the Gegenbauer reconstruction technique In [15], the Fourier spectral approximation was considered for discontinuous solutions of PDEs including the linear advection equations and inviscid Burgers' equation with the Gegenbauer reconstruction methods. The Gegenbauer post-processing technique has even been successfully applied to the weighted essentially non-oscillatory (WENO) approximation of a steady-state hyperbolic problem with a jump discontinuity [8] and recovered the desired order of accuracy in this local basis problem.

The question we wish to answer in this paper is whether the Gegenbauer post-processing technique can successfully recover order of accuracy in RBF approximations of discontinuous functions and RBF simulations of time-dependent linear and nonlinear PDEs with discontinuities.

3 Pseudospectral collocation methods for solution of time dependent PDEs

Any pseudospectral collocation method, including the RBF method, can be described as follows. Consider the one-dimensional hyperbolic conservation law

$$u_t + f(u)_x = 0, \quad x \in \Omega = [-1, 1], \quad u(x,t) : [-1,1] \times \mathbb{R}^+ \to \mathbb{R}, t > 0, \tag{1}$$

with the initial condition $u(x, 0) = g(x)$, and some appropriate boundary conditions.

We assume that we are given the numerical solution $U^n(x)$ at time $t = t^n$ (initially, this is the initial condition $U^0(x) = g(x)$) and we wish to move forward in time to approximate $U^{n+1}(x)$. As we are given $U^n(x)$ for all x, we therefore have the flux $f(U^n(x))$ at any points x we desire. We will need to differentiate $f(U(x))$ with respect to x, so we desire its expansion in the chosen basis functions $\psi_k(x)$,

$$f(U^n(x)) = \sum_{k=1}^{m} \theta_k \psi_k(x).$$

To find these using collocation, we chose a set of m points $\{x_j\}_{j=1}^m$ and we look for coefficients θ_k such that the expansion matches the function at these points

$$f(U^n(x_j)) = \sum_{k=1}^{m} \theta_k \psi_k(x_j) \qquad \forall x_j, \quad j = 1, ..., m.$$

This can be easily done by defining the interpolation matrix $\mathbf{M}$ whose elements are

$$M_{jk} = \psi_k(x_j),$$

the vector of function values $\mathbf{f}$ whose elements are $f(U^n(x_j))$, and a vector of unknown coefficients θ, and setting

$$\mathbf{M}\theta = \mathbf{f} \quad \rightarrow \quad \theta = \mathbf{M}^{-1}\mathbf{f}.$$

Once we have the expansion of $f(U^n(x))$, we can easily differentiate it,

$$f(U^n(x)) = \sum_{k=1}^{m} \theta_k \psi'_k(x)$$

where $\psi'_k(x)$ is the derivative of the basis functions with respect to x. Written in matrix form,

$$f(U^n)_x = \mathbf{D}\theta = \mathbf{D}\mathbf{M}^{-1}\mathbf{f},$$

where the differentiation matrix $\mathbf{D}$ is comprised of elements

$$D_{jk} = \left. \frac{\partial \psi_k(x)}{\partial x} \right|_{x_j} \qquad x_k, x_j \in \mathbf{X}.$$

To step the solution forward, we can use any high order method to solve the ODE system $\mathbf{U}' = -\mathbf{D}\mathbf{M}^{-1}\mathbf{f}(\mathbf{U})$.

4 Radial basis function methods

Given a one-dimensional input data point set, the center set $X = \{x_i | x_i \in \Omega, i = 1, \cdots, N\}$ in the domain $\Omega \subset \mathbb{R}$, the RBF interpolation of $f(x)$ denoted by $s_{f,X}(x)$ is then given by a linear combination of RBFs, $\psi_j(\| x - x_j \|, \epsilon_j)$, $j = 1, \cdots, N$;

$$s_{f,X}(x) = \sum_{j=1}^{N} \gamma_j \psi_j(\| x - x_j \|, \epsilon_j). \tag{2}$$

Here γ_j are the expansion coefficients and ϵ_j are the shape parameters, and $\| \cdot \|$ denotes the metric used (typically, the Euclidean norm $\| x - x_j \| = |x - x_j|$). Commonly used RBFs include

$$\sqrt{(x - x_j)^2 + \epsilon_j^2} \qquad \text{multi-quadric RBFs (MQ RBFs)} \tag{3}$$

$$\frac{1}{\sqrt{(x - x_j)^2 + \epsilon_j^2}} \quad \text{inverse multi-quadric RBFs (IMQ RBFs)} \tag{4}$$

$$\exp(-\epsilon_j (x - x_j)^2) \qquad \text{Gaussian RBFs} \tag{5}$$

The shape parameters ϵ_j are given or prescribed for the interpolation and the expansion coefficients γ_j are to be determined by the interpolation condition

$$s_{f,X}(x) = f(x), \quad x \in X, \tag{6}$$

which yields a linear system

$$\mathbf{M}\Gamma = \mathbf{f}, \tag{7}$$

where $\Gamma = (\gamma_1, \cdots, \gamma_N)^T$, $\mathbf{f} = (f(x_1), \cdots, f(x_N))^T$, and the interpolation matrix $\mathbf{M}$ is given by

$$M_{ij} = \psi_j(\| x_i - x_j \|, \epsilon_j). \tag{8}$$

The expansion coefficient vector Γ is obtained by solving Eq. (7)

$$\Gamma = \mathbf{M}^{-1}\mathbf{f},$$

where $\mathbf{M}$ is square and nonsingular [2].

The index j denotes the center point in the given X and the center set X is not necessarily structured, that is, it can have an arbitrary distribution. The arbitrary grid structure is one of the major differences between the RBF method and other global methods. Such a *mesh-free* grid structure yields higher flexibility especially when the domain is irregular.

The coefficients are determined by information from the entire domain, so the RBF method is a global method. Just as in the Fourier case we described above, the partial sum approximation with RBFs converges quickly if the function is smooth. If the function has a jump discontinuity, however, the rate of convergence deteriorates. This is the Gibbs phenomenon commonly observed with the global methods. The Gibbs phenomenon of the MQ-RBF method was investigated in our previous work [11], and it was shown that the Gibbs phenomenon is inevitable with evenly structured center set X and the non-zero shape parameters ϵ_i $\forall i$. It was also shown that the size of the over/under-shoots in the RBF approximation depends on the values of the shape parameters. For example, if $\epsilon_i \sim 1/N$, $\forall i$, the size of the oscillation is invariant for $\forall N$, while if ϵ_i, is fixed $\forall i$ and $\forall N$, the size of the oscillation increases with N. In [11], a shape-parameter adaptive method was proposed and it was shown that the shape parameters can be exploited to remove the Gibbs phenomenon in the MQ-RBF approximation of discontinuous functions. With the adaptive method, the shape parameters vanish at or near the local jump discontinuities. As the shape parameter vanishes, the local MQ-RBF functions become a piecewise linear function. By approximating the function with the linear function locally, the convergence away from the discontinuity can be considerably enhanced and the Gibbs oscillations are removed. While this method works well for the removal of the Gibbs phenomenon, it is difficult and expensive to implement in the case of a time-dependent PDE with a discontinuity. For this reason, we explore the Gegenbauer reconstruction technique for such problems. However, it is interesting that this adaptive method can be used for edge detection. In [4], the slow decay of the expansion coefficients was exploited to design a new edge detection method based on the shape parameter adaptive method with the MQ-RBF.

5 Gegenbauer reconstruction of RBF approximations of a discontinuous function

We begin with a RBF approximation

$$s_{f,X}(x) \equiv v(x) = \sum_{j=1}^{N} \gamma_j \psi_j(||x - x_j||, \epsilon_j), \quad x \in (-1,1), \quad x_j \in X$$

of a piecewise analytic function $u(x)$, $x \in (-1,1)$. The function $u(x)$ has some discontinuity in the domain $(-1,1)$, but is analytic in some interval of smoothness (a,b). Because $v(x)$ is a global approximation to $u(x)$ on $(-1,1)$, and $u(x)$ has a discontinuity in that interval, $v(x)$ will suffer from the Gibbs phenomenon.

To eliminate the Gibbs phenomenon, we post-process the approximation $v(x)$ to reconstruct $u(x)$ in the interval of smoothness (a, b), by the Gegebauer reconstruction method

$$w(x) = \sum_{l=0}^{m} g_l^\lambda G_l^\lambda(\xi(x)), \quad x \in (a, b)$$

where, as before, $\xi : x \in (a, b) \longrightarrow (-1, 1)$. There are two ways of computing this reconstructions, which we present below:

RBF-Fourier Gegenbauer reconstruction: The RBF-Fourier Gengenbauer reconstruction method is based on the Fourier-Gegenbauer reconstruction method which was originally developed for the reconstruction of the Fourier data. In this approach, the Gegenbauer coefficients are computed from the Fourier coefficients of the RBF approximation. The RBF approximation or interpolation is transformed to the Fourier space and then to the Gegenbauer polynomial space. The Fourier transformation of the RBF approximation $s_{f,X}(x), x \in (-1, 1)$ is given by

$$\hat{f}_k = \frac{1}{2}\int_{-1}^{1} s_{f,X}(x) e^{-ik\pi x} dx = \sum_{j=1}^{N} \gamma_j \frac{1}{2}\int_{-1}^{1} \psi_j(|x - x_j|, \epsilon_j) e^{-ik\pi x} dx,$$

where $\hat{f}_k$ are the Fourier coefficients of $s_{f,X}(x)$. We define the transformation matrix $\mathbf{W}$ which maps the RBF coefficients to the Fourier coefficients as follows

$$W_{kj} = \frac{1}{2}\int_{-1}^{1} \psi_j(|x - x_j|, \epsilon_j) e^{-ik\pi x} dx,$$

so that the Fourier coefficients are given by

$$\hat{\mathbf{f}} = \mathbf{W} \cdot \Gamma = \mathbf{W} \cdot \mathbf{M}^{-1} \cdot \mathbf{f},$$

where $\hat{\mathbf{f}} = (f_{-K}, \cdots, f_K)^T$ and $\mathbf{f} = (f(x_1), \cdots, f(x_N))^T$. If we define the transformation matrix $\tilde{\mathbf{W}}$ from the Fourier to Gegenbauer spaces, we have

$$\tilde{W}_{lk} = \frac{1}{h_l^\lambda}\int_{-1}^{1} (1 - \xi)^{\lambda - \frac{1}{2}} G_l^\lambda(\xi) \exp(ik\pi x(\xi)) d\xi, \quad x \in (a, b),$$

and we obtain

$$\mathbf{g} = \tilde{\mathbf{W}} \cdot \mathbf{W} \cdot \mathbf{M}^{-1} \cdot \mathbf{f},$$

where $\mathbf{g} = (g_0^\lambda, \cdots, g_N^\lambda)$.

Direct RBF-Gegenbauer reconstruction: The RBF-Gegenbauer method for $x \in (a, b)$ seeks the reconstruction $w(\xi(x))$ such that

$$w(\xi(x)) = \sum_{l=0}^{m} g_l^\lambda G_l^\lambda(\xi(x)),$$

where the expansion coefficients g_l^λ are *directly* determined from the local RBF approximation

$$g_l^\lambda = \frac{1}{h_l^\lambda} \int_{-1}^{1} (1-\xi)^{\lambda-\frac{1}{2}} G_l^\lambda(\xi) v(x(\xi)) d\xi.$$

Note that the values of $v(x)$ that we use are drawn only from the interval of smoothness (a, b). By the definition of $v(\xi(x))$ we have

$$g_l^\lambda = \sum_{l'=1}^{N} \gamma_{l'} \frac{1}{h_l^\lambda} \int_{-1}^{1} (1-\xi)^{\lambda-\frac{1}{2}} G_l^\lambda(\xi) \psi_{l'}(||x(\xi) - x_{l'}||, \epsilon_{l'}) d\xi.$$

Define the transformation matrix $\mathbf{H}$

$$H_{ll'} = \frac{1}{h_l^\lambda} \int_{-1}^{1} (1-\xi^2)^{\lambda-\frac{1}{2}} G_l^\lambda(\xi) \psi_{l'}(||x(\xi) - x_{l'}||, \epsilon_{l'}) d\xi,$$

where l runs from 0 to m and l' from 1 to N. Then the expansion coefficient vector $\mathbf{g}$ is given by

$$\mathbf{g} = \mathbf{H} \cdot \mathbf{\Gamma},$$

where again $\mathbf{\Gamma} = (\gamma_1, \cdots, \gamma_N)^T$. Then we obtain

$$\mathbf{g} = \mathbf{H} \cdot \mathbf{M}^{-1} \cdot \mathbf{f}.$$

For the Gegenabuer reconstruction theory to guarantee exponential convergence in the Fourier or polynomial cases, λ and m in the reconstruction must be proportional to N. We conjecture that this same constraint condition is required to recover high order convergence from the RBF approximation and as above, we will choose $\lambda = \alpha N, \quad m = \beta N$ with $0 < \alpha, \beta < 1$.

Since we assume that $u(x)$ is analytic in $x \in (a, b)$, $u(x)$ can be represented by the Gegenbauer polynomials such as

$$u(x) = \sum_{l=0}^{\infty} \hat{g}_l^\lambda G_l^\lambda(\xi(x)).$$

We define the error function $E(x) = |u(x) - w(x)|$ in $x \in (a, b)$, then the maximum norm of $E(x)$, $L_{\max}$ is given by

$$L_{\max} = || \sum_{l=0}^{m} \left(\hat{g}_l^\lambda - g_l^\lambda\right) G_l^\lambda(\xi(x)) + \sum_{l=m+1}^{\infty} \hat{g}_l^\lambda G_l^\lambda(\xi(x))||,$$
$$\leq || \sum_{l=0}^{m} \left(\hat{g}_l^\lambda - g_l^\lambda\right) G_l^\lambda(\xi(x))|| + || \sum_{l=m+1}^{\infty} \hat{g}_l^\lambda G_l^\lambda(\xi(x))||,$$

where let $||\cdot||$ denote the maximum norm. The second part of the RHS in the second inequality in the above equation is the regularization error. The regularization error decays fast with m because we assumed that $u(x)$ is analytic in $x \in (a, b)$. Then for the convergence of the Gegenbauer reconstruction of the RBF approximation, we only consider the truncation error, i.e. the first part of the RHS, $|| \sum_{l=0}^{m} \left(\hat{g}_l^\lambda - g_l^\lambda\right) C_l^\lambda(\xi(x))||$. The truncation error (TE) is then

$$TE = || \sum_{l=0}^{m} \left(\hat{g}_l^\lambda - g_l^\lambda\right) C_l^\lambda(\xi(x))||$$
$$\leq C_m^\lambda(1) ||\hat{\mathbf{g}} - \mathbf{g}||_1,$$

where $||\cdot||$ denotes the matrix 1-norm and we use $\max_{-1\leq\xi\leq1} C_l^\lambda(\xi) = C_m^\lambda(1)$ for $\forall l = 0, \cdots, m$ [1]. And $C_m^\lambda(1) = \frac{\Gamma(m+2\lambda)}{\Gamma(2\lambda)m!}$. For example, if the following can decay uniformly with N with the proper choices of α and β for $\lambda = \alpha N$ and $m = \beta N$, the direct RBF-Gegenbauer reconstruction method recovers the accuracy from the oscillatory RBF approximation

$$\frac{\Gamma(m+2\lambda)}{\Gamma(2\lambda)m!} ||\hat{\mathbf{g}} - \mathbf{H} \cdot \mathbf{M}^{-1} \cdot \mathbf{f}||_1.$$

Here note that $\mathbf{f}$ is defined in the entire domain Ω and the information of $u(x)$ in $x \in (a, b)$ is only partially contained in $\mathbf{f}$. It is not obvious that the above term can decay exponentially. The detailed analysis and investigation should be left for our future work at this moment. The numerical results shown in the following sections, however, indicate that the Gegenbauer reconstruction method indeed recovers the exponential convergence from the oscillatory RBF approximations. Also, the numerical experiments show that the RBF-Fourier Gegenbauer method and the direct RBF-Gegenbauer method yield the similar results.

6 Numerical Results

In this section, we provide the numerical evidence that the RBF-Gegenbauer method does indeed recover high order from the RBF approximation of discontinuous functions. For the numerical tests, we use both the piecewise analytic function and the numerical solutions of hyperbolic equations which contains the local discontinuities.

In the following numerical experiments we use the MQ basis functions for ψ, which yields the interpolation matrix

$$M_{ij} = \sqrt{(x_i - x_j)^2 + \epsilon_j^2}, \quad i, j = 1, \cdots, N, \tag{9}$$

and the differentiation matrix

$$D_{ij} = \frac{x_i - x_j}{\sqrt{(x_i - x_j)^2 + {\epsilon_j}^2}}, \quad i, j = 1, \cdots, N, \tag{10}$$

6.1 Reconstruction of piecewise analytic function

We consider the following piecewise analytic function $f(x) \in \mathbb{R}$ for $x \in [-1, 1]$,

$$f(x) = \begin{cases} \sin(x) & -1 \leq x \leq 0 \\ \cos(x) & 0 < x \leq 1. \end{cases} \tag{11}$$

A local discontinuity exists at $x = 0$ and its jump magnitude $[f] = 1$. The same problem has been considered in [11] where the Gibbs phenomenon with MQ-RBFs for $f(x)$ with N has been investigated. The interpolation of $f(x)$ with the MQ-RBF yields the spurious oscillations near $x = 0$ which do not disappear even as N increases. In [11], it has been shown that the uniform convergence is not obtained for such function due to the discontinuity at $x = 0$. The L_∞ error does not decay at $x = 0$ and overall convergence away from $x = 0$ is slow.

Table 1 shows the L_∞ errors in RBF approximation of $f(x)$ without and with the Gegenbauer post-processing. For the Gegenbauer reconstruction, we use the direct Gegenbauer-RBF method, with $m = \lambda = \frac{N}{8}$.

As the table indicates, because of the Gibbs phenomenon, the L_∞ error of the RBF approximation does not decay with N. However, after reconstruction with the Gegenbauer method, the L_∞ error of the RBF-Gegenbauer approximation decays rapidly.

6.2 Linear advection equation

Consider a time-dependent linear advection equation with variable coefficient

Table 1
The L_∞ errors of the RBF approximation before and after the reconstruction with $\epsilon = 0.1, \forall N$. For the reconstruction, $\lambda = m = \frac{N}{8}$.

N	Before reconstruction	After reconstruction
16	4.3821(-1)	1.4768(-2)
32	4.2036(-1)	4.5648(-3)
64	3.5943(-1)	3.0938(-5)
128	2.2678(-1)	4.1430(-7)

$(-n) \equiv 10^{-n}$

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0, \quad u : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}, \quad x \in \Omega = [-1, 1], \quad t > 0, \tag{12}$$

with the initial condition

$$u(x, 0) = \begin{cases} 1, \; x \in [-0.5, 0.5] \\ 0, \quad \text{otherwise} \end{cases}. \tag{13}$$

Note that the boundary conditions at $x = \pm 1$ are not required with Eq. (12) because the solution advects out of the domain at both boundaries. It is now well known that the accuracy of the RBF approximation is degraded due to the boundary conditions for differential equations, and currently there is no consistent boundary treatment for the RBF method resolving this problem. Numerical results found in the literature show that the best L_∞ errors are about $10^{-4} \sim 10^{-5}$ even after a short time integration, for example in [14]. As the final time increases, the errors due to the boundary conditions also increase. Since we are interested in the reconstruction of the RBF approximation contaminated only by the Gibbs phenomenon due to the discontinuity, we try to minimize the errors due to the boundary conditions by considering boundary free equations such as Eq. (12). The exact solution of Eq. (12) at the time t is given by

$$u(x, t) = \begin{cases} 1, \; x \in [-x_c, x_c] \\ 0, \quad \text{otherwise} \end{cases}, \tag{14}$$

where the right front of discontinuity x_c is given by $x_c(t) = 0.5\exp(t), t \geq 0$. For the numerical experiments, we use $N = 16, 32, 46, 64,$ and 76 with the equidistant centers. The time step dt is $dt = 0.4dx$ with $dx = \frac{2}{N-1}$. We convect the equation from $t = 0$ to final time $t_f = 0.345$ for which the right shock front is located at $x_c \sim 0.706$. For the time integration we use the TVD 3rd order Runge-Kutta scheme. The RBF approximation is very oscillatory containing the Gibbs oscillations from $t = 0$. The oscillations advect with the variable speed x toward the boundaries. We post-process this solution using the Gegenbauer method only in the middle portion of the solution in $x \in [-x_c, x_c]$ between the two shock fronts. That is, we assume that we know the exact shock locations. For the Gegenbauer reconstruction we use $m = \frac{N}{4}$ and $\lambda = \frac{N}{4}$. The RBF solution is mapped in $\xi = [-1, 1]$, $\xi = \frac{1}{x_c}(x - x_c) + 1$. Figure 1 shows the numerical solution (represented in blue line with open circle) with the RBF method and the exact solution (represented in red solid line) at $t = t_f$. The Gegenbauer reconstructed solution is also shown in a black solid line, which is overlapped with the exact solution almost everywhere. As shown in the figure, the RBF solution is oscillatory over the whole domain. Figure 2 shows the pointwise errors with various N before the Gegenbauer reconstruction method is applied (left) and the pointwise errors after the Gegenbauer reconstruction method is applied (right) for $x \in [-x_c, x_c]$. Note that the $x-$axis of the figure is given in the unit space $\xi(x)$. Each solid line represents the pointwise errors with different N. The right figure clearly shows which line corresponds which N. In the right figure, each line corresponds to $N = 16, 32, 46, 64$, and 76 from top to bottom, respectively. Before the Gegenbauer reconstruction is applied, the errors do not decay with N (Figure 2, left). After the Gegenbauer reconstruction method is applied, it is clearly seen that the high order convergence is recovered (Figure 2, right). Here notice that the pointwise errors are shown in different scale for the left and right figures.

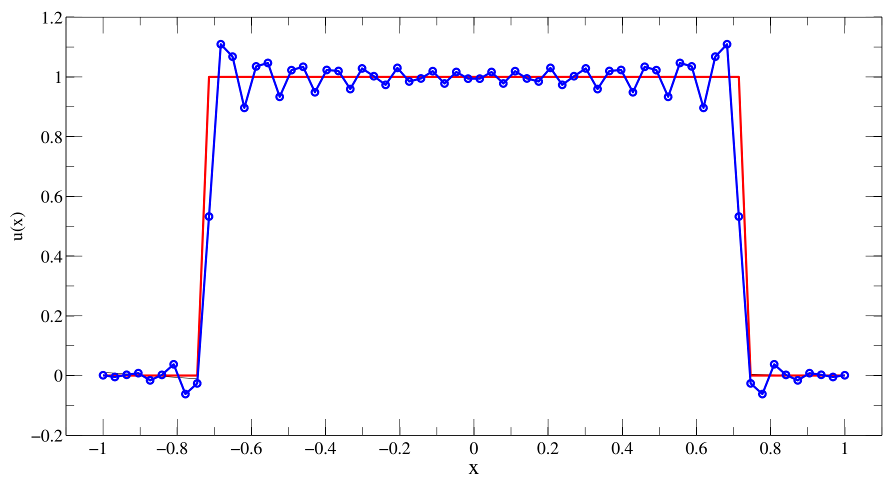

Fig. 1. The RBF solution (oscillatory blue solid line with open circle) and the exact solution (red solid line) of Eq. 12 at $t = 0.345$ with $N = 64$. The reconstructed solution with the Gegenbauer method is shown as a black solid line as well but is overlapped with the exact solution almost everywhere.

Figure 3 shows the decay of the L_∞ errors with N. The red line represents the errors of RBF approximations without the Gegenbauer reconstruction method and the blue line with the RBF-Gegenbauer reconstruction method. As shown in the figure, the high order convergence has been successfully recovered with

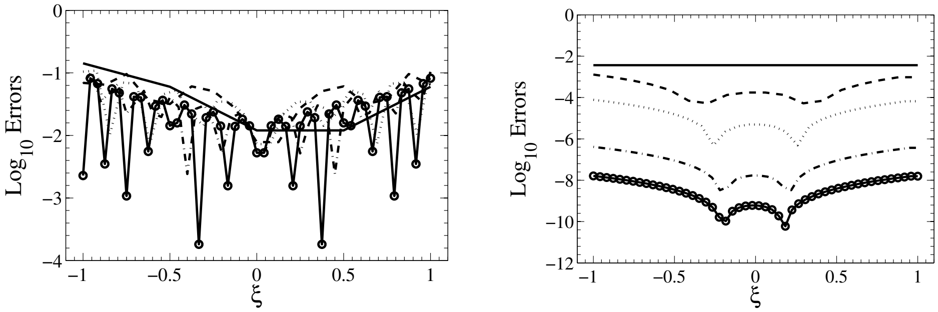

Fig. 2. Left: The pointwise errors of the RBF approximation of Eq. (12) in logarithmic scale. Right: The pointwise errors after the Gegenbauer reconstruction method is applied. $t_f = 0.345$, $N = 16, 32, 46, 64$, and 76 and $dt = 0.4\frac{2}{N-1}$. Note that the errors are plotted with ξ. The right figure clearly shows which line corresponds to which N. Each line is corresponding to $N = 16, 32, 46, 64$, and 76 top to bottom respectively in the right figure.

the Gegenbauer method.

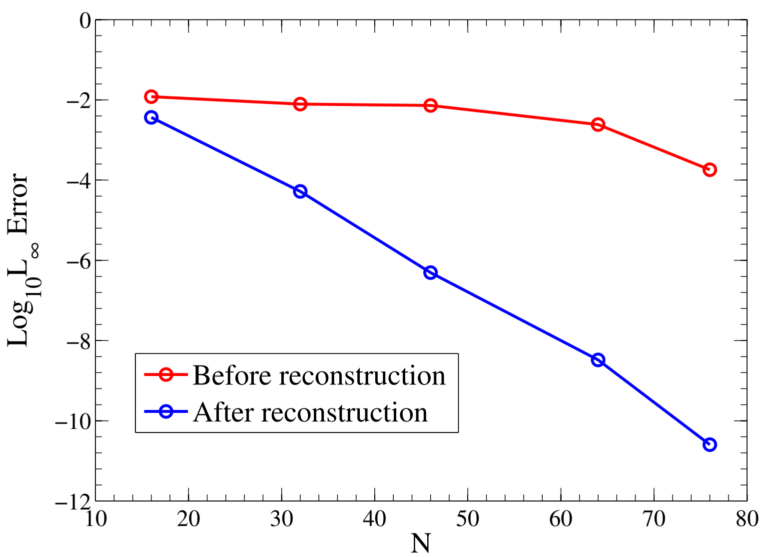

Fig. 3. Convergence of L_∞ errors with N. The red and blue lines indicate the L_∞ errors of the RBF approximations without and with the Gegenbauer reconstruction method, respectively.

6.3 Burgers' equation

Now we consider the Burgers' equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}, \quad x \in \Omega = [-1,1], \quad t > 0, \quad (15)$$

with the initial condition

$$u(x,0) = -\sin(\frac{3}{2}\pi x). \tag{16}$$

This problem requires no boundary conditions since the solution u leaves the domain at both boundaries. The shock forms at $x = 0$ at $t = \frac{2}{3\pi}$. Due to the symmetry around $x = 0$, the shock does not move once formed.

Again we assume that we know the shock location for all time t if the shock exits. In general, the shock location is unknown and should be found first before applying the Gegenbauer reconstruction method using such techniques as [4]. However, the objective of the paper is to show the recovery of high order convergence, so this initial condition is chosen to simplify matters.

Table 2 shows the values of m and λ used for the Gegenbauer reconstruction. These values were found by numerical experiments. There is no systematic way to find the optimal values of m and λ for N. In [7], it was discussed that the errors with m and λ are not necessarily convex and can have multiple local minima. In general, it is not easy to find the optimal values of m and λ minimizing the errors and the trial-and-error approach is used.

Table 2
The values of λ and m for the Gegenbauer reconstruction.

N	post-processing	in situ
30	$\lambda = 2 \quad m = 2$	$\lambda = 2 \quad m = 3$
70	$\lambda = 3 \quad m = 4$	$\lambda = 4 \quad m = 6$
130	$\lambda = 6 \quad m = 7$	$\lambda = 8 \quad m = 12$

For the numerical example, we have chosen such final time so as to allow us to compare the post-processed solution to the RBF solution. If we step much further, the RBF solution blows up and we have no basis for comparison.

Figure 4 shows the RBF approximations of Eq. (15) at $t = 0.4$ with $\epsilon = 0.1$ (left) and $\epsilon = 0.1/N$ (right) for $N = 70$. It is interesting to observe that the Gibbs oscillations are predominantly near $x = -0.5$ and $x = 0.5$ for the case of $\epsilon = 0.1$, but near the shock at $x = 0$ for $\epsilon = 0.1/N$. This interesting phenomenon will be investigated in our future research work. Although the locations of the Gibb oscillations are different depending on the values of the shape parameters ϵ, these oscillations do not decay with t and N. Thus the RBF method does not have uniform convergence in the presence of a shock.

For comparison more in detail, the solutions at $t = t_f = 0.4$ are considered for

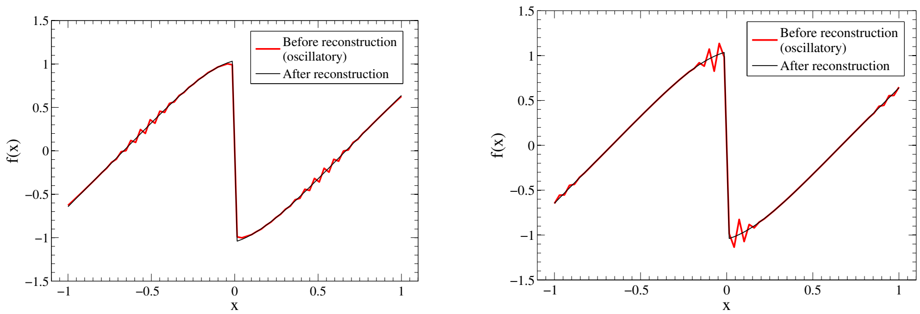

Fig. 4. RBF approximations of Eq. 15 before and after the Gegenbauer reconstruction. Left: $\epsilon = 0.1$. Right: $\epsilon = 0.1/N$. $N = 70$, $m = 4$, and $\lambda = 3$.

different cases. For the numerical experiments, we also consider the Gegenbauer reconstruction method in situ. The Gegenbauer reconstruction in situ utilizes the Gegenbauer post-processing technique during the time integration. The usual post-processing technique is applied once at the final time. The Gegenbauer method in situ applies the post-processing technique whenever it is needed to stabilize the solution. For the Gegenbauer method in situ, the nonsmoothness detection technique is needed to determine when the post-processing should be used. For the numerical example, a simple first order edge detection method explained briefly in below. The details of the Gegenbauer method in situ is beyond the main purpose of this paper and will be discussed in our future research.

Figure 5 shows the pointwise errors with various N of the RBF approximations at $t = t_f$ before and after the Gegenbauer reconstruction. A fixed shape parameter $\epsilon = 0.1, \forall N$ is used for the consistency with the other numerical examples in this paper. The left figure shows the pointwise errors without the Gegenbauer post-processing, the middle figure with the Gegenbauer post-processing applied once at $t = t_f$ and the right figure with the Gegenbauer post-processing in situ. For the Gegenbauer method in situ, the Gegenbauer reconstruction has been done three times before $t = t_f$. The same CFL condition is used $dt = 0.4dx$ as for Eq. 12. The reconstruction has been carried out piecewise, such that two reconstructions are obtained in $x = [-1, 0]$ and $x = [0, 1]$ respectively. As the left figure shows in Figure 4, the Gibbs oscillations are dominated near $x = -0.5$ and $x = 0.5$ but not $x = 0$. For this reason, in the left figure of Figure 5, the pointwise errors of the RBF approximations without the Gegenbauer reconstruction do not converge with N near $x = -0.5$ and $x = 0.5$. After the Gegenbauer reconstruction is applied, the pointwise errors near $x = -0.5$ and $x = 0.5$ decay nicely as the middle and right figures of Figure 4 show. The Gegenbauer reconstruction in situ shows the best result among them. This result indicates that the Gegenbauer reconstruction method in situ would be a good candidate for stabilizing the RBF approximations for discontinuous problems.

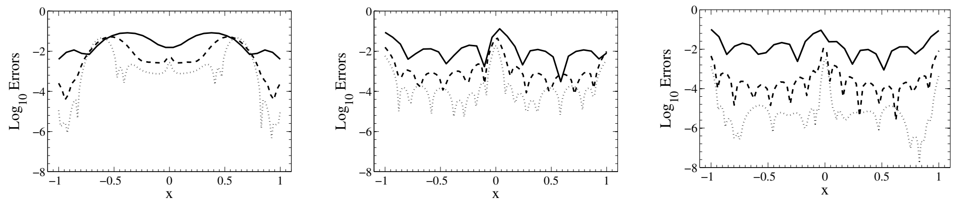

Fig. 5. Pointwise errors in logarithmic scale with $N = 30$ (solid line), 70 (long dotted line) and 130 (dotted line). Left: Without the Gegenbauer post-processing. Middle: With the Gegenbauer post-processing once at $t = t_f$. Right: With the Gegenbauer post-processing in situ.

Figure 6 shows the L_2 (left figure) and L_∞ (right figure) errors in logarithmic scale. The symbols $\triangle$, $\square$ and $\circ$ denote the errors with the RBF approximations without the Gegenbauer post-processing, with the Gegenbauer post-processing once at t_f and with the Gegenbauer post-processing in situ, respectively. The errors are calculated at the collocation points in $x = [-1, 1]$. To avoid the boundary effects of the Gegenbauer method due to the round-off errors near the boundaries, few boundary points are excluded [7]. Since $x = 0$ is the boundary for each Gegenbauer reconstruction, few collocation points are also excluded near $x = 0$. For $N = 30, 70$, and 130, 2, 5, and 9 points are excluded respectively, at $x = -1, 1, 0$. The figures show that the Gegenbauer reconstruction improves the error behavior significantly although the errors with the RBF approximation do not converge. The figure also shows that the Gegenbauer reconstruction in situ yields the best result among them.

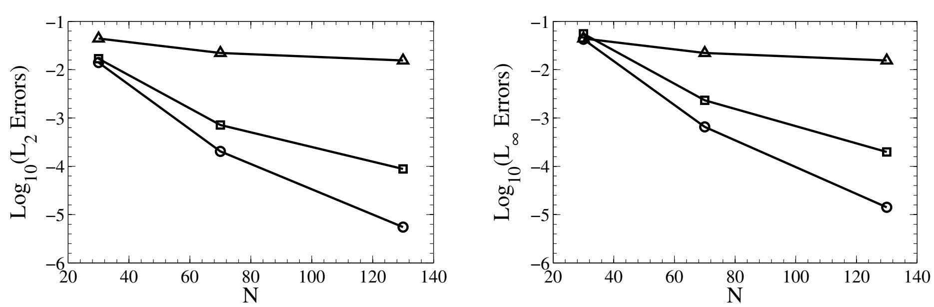

Fig. 6. Convergence of L_2 (Left) and L_∞ (Right) errors in logarithmic scale with N. The symbols $\triangle$, $\square$ and $\circ$ denote the errors with the RBF approximations without the Gegenbauer post-processing, with the Gegenbauer post-processing once at t_f and with the Gegenbauer post-processing in situ, respectively.

Table 3 shows the L_∞ errors for different cases in Figure 6. The L_2 errors are given in logarithmic scale in the table and the orders of convergence are also given in parenthesis for $N = 70$ and $N = 130$. As shown in the table, the order of convergence has been significantly improved if the solution is post-processed both once at $t = t_f$ and in situ. For the post-processing once at $t = t_f$ recovers about the 4th order convergence and the post-processing in situ about the 5th and 6th order convergence. If $N > 130$, the interpolation matrix is highly illconditioned and the boundary effects become dominant in the domain and the L_2 errors do not decay beyond 10^{-5} which is the typical error barrier found in many RBF literatures for hyperbolic problems. Consequently the order of convergence is not improved even if N becomes larger than $N = 130$.

Table 3
L_∞ errors in logarithmic scale for the RBF approximation, the post-processed solution at $t = t_f$, and the post-processed solution including post-processing in situ. The numbers in the parenthesis for $N = 70$ and 130 denote the order of convergence.

N	RBF approximation	post-processing at end	post-processing in situ
30	-1.257	-1.354	-1.371
70	-1.654 (1.0789)	-2.635 (3.4812)	-3.185 (4.9297)
130	-1.809 (0.5765)	-3.704 (3.9763)	-4.849 (6.1894)

Typically, in pseudospectral and spectral methods simulations of nonlinear PDEs with shocks, the method is stabilized by adding viscosity or by filtering. For RBF simulations, these techniques are not yet well developed. However, the Gegenbauer post-processing process can be applied to the RBF method for stabilization before the final time. We tested this approach of *post-processing in situ* and found that for this simple problem it adequately stabilizes the method and improves the accuracy as well. To perform post-processing in situ, we devised a simple measure of oscillations, and activated the post-processor when a threshold limit was exceeded. The oscillation measure relies on the centered difference

$$\Delta_n^2 = \frac{U_{n+1} - 2U_n + U_{n-1}}{\Delta x}.$$

If the L_2 norm of this was above a certain threshold, post-processing in situ was performed. Table 3 shows the convergence using post-processing in situ. The post-processing in situ will be further discussed in our future work. The numerical examples presented here indicate that the RBF approximation for discontinuous problems can successfully recover the high order convergence once a proper post-processing technique is applied such as the Gegenbauer reconstruction method.

7 Concluding remarks

In this paper, we presented the numerical evidence which demonstrates that the Gegenbauer reconstruction method can recover high order of accuracy in

RBF approximations of discontinuous functions. First using the piecewise analytic function, we demonstrated that although the maximum-norm errors of the RBF approximation do not decay as the number of points is increases, the maximum-norm errors of the Gegenbauer-reconstructed approximation decay rapidly. We also considered the numerical solutions of linear and nonlinear hyperbolic PDEs. We showed that the RBF-Gegenbauer method recovers high order convergence in the numerical solution in time domain. The significance of these results is two-fold. First, they verify that although high order information seems to be lost in RBF approximations if the function to be approximated has a local discontinuity, it can be recovered using an appropriate post-processing technique. Second, they show that the Gegenbauer polynomials are an appropriate basis for such reconstruction. This suggests that the Gegenbauer basis is *Gibbs complementary* to MQ-RBF basis functions.

RBF approximations have been developed and proven useful for complex problems with difficult geometries. Their mesh-free properties and ease of their implementation make them favored among scientists working in atmospheric studies [6]. A technique which would allow RBFs to overcome the Gibbs phenomenon would allow us to extend the range of applicability of RBFs to problems with discontinuities.

Our future work will center around several questions such as the optimum values of m and λ for convergence, the development of a rigorous proof for the recovery of convergence, the dependence of the locations of the oscillations for time dependent RBF solutions on different shape parameters when a local jump exists, and the stabilizing technique based on the Gegenbauer post-processing technique for nonlinear time-dependent problems, i.e. the Gegenbauer post-processing in situ.

Acknowledgements

This work has been supported by the NSF under Grant No. DMS-0608844. The authors also acknowledge useful communications with David Gottlieb.

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