Homework 3

due of 2nd class of week 8

Problem 1: (integration by parts) (3 points)
Find the indefinite integral:
\[ \int (3x^2+1)\ln(x^2+1)\,dx \]
(hint: just choose the appropriate \( u' \), \( v \) then write out every details and things will work like a charm; also notice that while simplifying the 2nd integral, you better write \( x^3 + x = x(x^2 + 1) \) to see some cancellation)

Solution:
We use integration by parts:
\[ u' = 3x^2 + 1, \quad v = \ln(x^2 + 1) \]
\[ u = x^3 + x, \quad v' = \frac{2x}{x^2 + 1} \]
then
\[ \int (3x^2+1)\ln(x^2+1)\,dx = (x^3 + x)\ln(x^2 + 1) - \int (x^3 + x) \frac{2x}{x^2 + 1} \,dx \]

Like the hint says \( x^3 + x = x(x^2 + 1) \), we can work out the 2nd integral as following:
\[ \int x(x^2+1) \frac{2x}{x^2 + 1} \,dx = \int 2x^2 \,dx = \frac{2}{3}x^3 + C \]
Which means
\[ \int (3x^2+1)\ln(x^2+1)\,dx = (x^3 + x)\ln(x^2 + 1) - \frac{2}{3}x^3 + C \]
(QED)

Problem 2: (definite integrals) (2 points)
a) Use your answer in problem 1, compute the definite integral:
\[ \int_{0}^{1} (3x^2+1)\ln(x^2+1)\,dx \]
b) Use Simpson's rule (the simple version) to estimate the value of the above definite integral, find the error between this approximation result and the exact result from problem 2a.

Solution:
a) Using the answer of problem 1, we have:
\[ \int_{0}^{1} (3x^2+1)\ln(x^2+1)\,dx = (x^3 + x)\ln(x^2 + 1) - \frac{2}{3}x^3 \bigg|_{0}^{1} = [(1+1)\ln(1+1) - \frac{2}{3}]-[0-0] \]
\[ = 2\ln 2 - \frac{2}{3} \approx 0.7196 \]
b) The Simpson's rule says:
\[ I \approx \frac{1}{6} [ f(0) + 4f\left(\frac{1}{2}\right) + f(1)] = \frac{1}{6} [ 1\ln 1 + 4 \cdot \left(\frac{3}{4}\right)\ln(1+1) + 4\ln 2 ] \]
\[ = \frac{1}{6} (7\ln \frac{5}{4} + 4\ln 2) \approx 0.7224 \]
The error of this approximate answer is error = 0.7224 - 0.7196 = 0.0028 (quite small)
Problem 3: (integration by partial fractions) (5 points)
Compute the indefinite integral
\[ \int \frac{x^2 - 5x + 14}{x(x-1)^2(x-2)} \, dx \]

Solution
Setup partial fractions:
\[
\frac{x^2 - 5x + 14}{x(x-1)^2(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{x-2}
\]
Want:
\[
x^2 - 5x + 14 = A(x-1)^2(x-2) + B(x)(x-1)(x-2) + C(x)(x-2) + D(x)(x-1)^2
\]
Plug in \( x = 0 \) yields
\[ 14 = A(-1)^2(-2) = -2A \quad \text{which means} \ A = -7 \]
Plug in \( x = 1 \) yields
\[ 1 - 5 + 14 = C(1)(1-2) = -C \quad \text{which means} \ C = -10 \]
Plug in \( x = 2 \) yields
\[ 4 - 10 + 14 = D(2)(2-1)^2 = 2D \quad \text{which means} \ D = 4 \]
Lastly, plug in \( x = 3 \) yields
\[ 9 - 15 + 14 = A(3-1)^2(3-2) + B(3)(3-1)(3-2) + C(3)(3-2) + D(3)(3-1)^2 = 4A + 6B + 3C + 12D \]
which means
\[ 8 = 4(-7) + 6B + 3(-10) + 12(4) = 6B - 10 \quad \Rightarrow 6B = 18 \quad \Rightarrow B = 3 \]
Therefore,
\[
\int \frac{x^2 - 5x + 14}{x(x-1)^2(x-2)} \, dx = \int \left[ \frac{-7}{x} + \frac{3}{x-1} + \frac{-10}{(x-1)^2} + \frac{4}{x-2} \right] \, dx
\]
\[ = -7 \ln|x| + 3 \ln|x-1| - \frac{10}{x-1} + 4 \ln|x-2| + C \]