Homework 4

Due on 2nd class of week 11
Due on 1st class of week 12

Problem 1: (integral solution) (2 points)
Find the solution of the differential equation
\[ y' = t^5 + 2t - 1 \]
that satisfies the initial condition:
\[ y(0) = 1 \]

Solution:
There is no \( y \) on the RHS, we can just integrate the equation and get:
\[ y(t) = \int (t^5 + 2t - 1) \, dt = \frac{t^6}{6} + t^2 - t + C \]
The initial condition is \( y(0) = 1 \), which implies \( C = 1 \).
Answer:
\[ y(t) = \frac{t^6}{6} + t^2 - t + 1 \]

Problem 2: (checking solution) (3 points)
Given the following function:
\[ y = f(t) = e^{2t} - t^2 + 1 \]
a) Prove that the above function is one solution of the following differential equation:
\[ y' = 2(y + t^2 - t - 1) \]
b) Show that the given function is one solution of the differential equation:
\[ y' = 2(e^{2t} - t) \]
c) Is it true that the given function is the solution of the following differential equation:
\[ (y' + 2t)^2 = 4e^{2t}(y + t^2 - 1) \]

Solution:
Just compute \( f'(t) = 2e^{2t} - 2t \) and plug in each equation:
a) RHS = \( 2((e^{2t} - t^2 + 1) + t^2 - t - 1) = 2e^{2t} - 2t \) \( \rightarrow \) TRUE
b) RHS = \( 2e^{2t} - 2t \) \( \rightarrow \) TRUE
c) YES. Because
\[ \text{LHS} = \left[(2e^{2t} - 2t) + 2t\right]^2 = [2e^{2t}]^2 = 4e^{4t} \]
\[ \text{RHS} = 4e^{2t}[(e^{2t} - t^2 + 1) + t^2 - 1] = 4e^{2t}e^{2t} = 4e^{4t} = \text{LHS} \]

Problem 3: (separation of variables) (5 points)
Consider the differential equation:
\[ y' = \frac{3t^2 + 4t}{4y^3} \]
a) Solve this equation using separation of variables.
b) Find the solution that satisfies the initial data: \( y(0) = -2 \)

Solution
a) \( \frac{dy}{dt} = \frac{3t^2 + 4t}{4y^3} \) \Rightarrow \int 4y^3 \, dy = \int (3t^2 + 4t) \, dt \Rightarrow y^4 = t^3 + 2t^2 + C \\

We have \( y = \pm \sqrt[4]{t^3 + 2t^2 + C} \)

b) If \( y(0) = -2 \), we should choose the (−) sign, and then plug in \( t = 0 \):

\[-2 = -\sqrt[4]{0^3 + 2(0)^2 + C} \Rightarrow C = 16\]

Therefore, the solution is \( y(t) = -\sqrt[4]{t^3 + 2t^2 + 16} \)