Chapter 5
Integration

Anti-derivative
A function \( F(x) \) is called anti-derivative of a function \( f(x) \) if
\[
F'(x) = f(x)
\]
Note: if \( f(x) \) is a continuous function, it has anti-derivative. Anti-derivative is not unique, we could notice that if \( F(x) \) is an anti-derivative of \( f(x) \) then \( G(x) = F(x) + C \) is another anti-derivative of \( f(x) \), for:
\[
G'(x) = \left[ F'(x) + C \right]' = F'(x) = f(x), \quad C \text{ is some constant}
\]
In fact, all anti-derivative of a function \( f(x) \) are different in a constant.

Some common anti-derivatives:

<table>
<thead>
<tr>
<th>( F(x) )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{x^{n+1}}{n+1} )</td>
<td>( x^n ), ( n \neq -1 ) (power rule)</td>
</tr>
<tr>
<td>( c \cdot x )</td>
<td>( c )</td>
</tr>
<tr>
<td>( \frac{-1}{(n-1)x^{n-1}} )</td>
<td>( \frac{1}{x^n} ) (( = x^{-n} ) ) for ( n \neq 1 )</td>
</tr>
<tr>
<td>( \frac{n}{n+1} \sqrt[n+1]{x^{n+1}} )</td>
<td>( \frac{\sqrt[n]{x}}{x^{1/n}} ) (( = x^{1/n} ) )</td>
</tr>
<tr>
<td>( \frac{2}{3}x^{3/2} )</td>
<td>( \sqrt{x} )</td>
</tr>
<tr>
<td>( -\frac{1}{x} )</td>
<td>( \frac{1}{x^2} )</td>
</tr>
</tbody>
</table>

And many more that you could make your own list. Notice that all of the above are simply derived from the power rule. Therefore, it’s best that you memorize and master only the very first one.

Following are the anti-derivative for exponential and logarithmic function, that you should be familiar already.

<table>
<thead>
<tr>
<th>( e^x )</th>
<th>( e^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{a} e^{ax} )</td>
<td>( e^{ax} )</td>
</tr>
<tr>
<td>( \ln x )</td>
<td>( \frac{1}{x} )</td>
</tr>
<tr>
<td>( \frac{1}{a} \ln x )</td>
<td>( \frac{1}{ax} )</td>
</tr>
</tbody>
</table>
**Indefinite Integral**

If \( F(x) \) is an anti-derivative of a function \( f(x) \), we can write

\[
\int f(x) \, dx = F(x) + C
\]

Notice that the “ \( dx \) ” and “ \( +C \)” are just parts of the notation. We don't specify \( C = 5 \) or \( C = -11 \) or anything. On the other hand, if the variable is \( y, z, t, ... \) instead of \( x \), we will write \( dy, dz, dt, ... \) respectively.

**Properties:**

1. \( \int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx \)
2. \( \int c \cdot f(x) \, dx = c \int f(x) \, dx \), \( c \) is some constant

**Integration by substitution**

Introduce a new variable \( u \) as a function of \( x \):

1. substitution: \( u = u(x), \, du = u'(x) \, dx \)
2. then apply:

\[
\int f(u(x)) \, u'(x) \, dx = \int f(u) \, du
\]

Notice: you need a lot of examples to understand this technique. Refer to lecture and exercises.

**Definite Integral and Riemann summation**

\[
\int_{a}^{b} f(x) \, dx
\]

\( \text{where } a \text{ and } b \text{ are two constants} \)

I introduced in lecture the Riemann summation to define as well as compute/approximate an definite integral. However, we will not focus on using this, for the fact that it involved way too much computation (by hands!!!). Throughout the course, we will compute definite integral conveniently by another technique, using the Fundamental Calculus theorem:

\[
\int_{a}^{b} f(x) \, dx = F(x)\big|_{a}^{b} = F(b) - F(a)
\]

if \( f(x) \) is a continuous function.

**Topics:**

**Average value of a function:**

The average value of a function \( f(x) \) on an interval \((a, b)\) can be defined as:

\[
\frac{1}{b-a} \int_{a}^{b} f(x) \, dx
\]

**Area under a curve:**

The area under a positive valued curve (and above x axis) in a boundary from \( a \) to \( b \) can be computed as:

\[
\text{Area} = \int_{a}^{b} f(x) \, dx
\]

In general, the area bounded by a general curve and x-axis, bounded left and right by \( x = a \) and \( x = b \) is:

\[
\text{Area} = \int_{a}^{b} |f(x)| \, dx
\]
Area between two curves:
The area below the graph of a function \( f(x) \) and above the graph of another function \( g(x) \), bounded by \( x = a \) and \( x = b \) is:

\[
\text{Area} = \int_{a}^{b} \{ f(x) - g(x) \} \, dx
\]

In many cases, \( x = a \) and \( x = b \) are the x-coordinates of the two intersections of the two curves. In general, if there is no specified about upper and lower curves, we can compute the area as:

\[
\text{Area} = \int_{a}^{b} \left| f(x) - g(x) \right| \, dx
\]