Test One

September 23 2008

Your name:
Student #:

Important:
  x Show your work. A sole answer without anything to show that you honestly solve the problem will be graded zero.
  x You can use a calculator, but only to help you in calculating or checking your work. Simply copying answer from calculator will be considered cheating (for example, input a quadratic equation in a scientific calculator and copy its solution).
  x You can use a print of the online lecture note, NOT YOURS, so it can be considered fair for all students. Along with the note, you can also bring a small cheat-sheet (not a A4 sized one).
  x I DO give partial credits, at my standard. This means, writing some weird stuff and claim it solved the question doesn’t mean you will get some credits.
  x I prefer you give answer in rational value or radical form, a decimal answer (like, 2.6458 in stead of $\sqrt{7}$ ) is ugly and in some situation will not be acceptable.
  x No make-up test !!! except super extraordinary reason. Don’t forget to write your name on the first page of your work, or on this sheet and staple to your papers. Also remember to write your name on the back of the last sheet which helps me in returning your tests later.

Test questions:

Problem 1: 5 points
Find the domain of the following function:

$$f(x) = \frac{1}{x-4} + \sqrt{5-x}$$

Solution:
We need $x - 4 \neq 0$ and $5 - x \geq 0$, which means $x \leq 5$ and $x \neq 4$

Problem 2: 5 points
You are going to make a rectangle and a square from a 100 feet colorful string. The shapes are described as below. Suppose the rectangle is about to be created first, with size $x$ and $3x$.
Express the area of the square as a function of $x$.

If the size of the rectangle is 10x30 (that is, $x = 10$), how big is the square (what is its area)?

Solution:
The perimeter of the rectangle is $2(x + 3x) = 8x$, hence what’s left to make the square is $4y = 100 - 8x$, which yields $y = 25 - 2x$. This leads to the expression of the area of the square: $f(x) = (25 - 2x)^2$. When $x = 10$, we have area = $(25 - 2 \cdot 10)^2 = 5^2 = 25$
Problem 3: 5 points
You are given a quadratic function: \( f(x) = 4x^2 - 16x + 15 \) and its graph is a parabola (P)

[a] (1 point) Find the \( y \)-intercept of P and y axis

[b] (2 points) Find the two \( x \)-intercepts of P and x axis (that means, solve \( f(x) = 0 \))

[c] (2 points) Find the equation of the line which goes through the vertex of P and is parallel with the line \( y = 4x \)

Solution:

[a] \( y \)-intercept : \((0, 15)\)

[b] \( \Delta = (-16)^2 - 4 \cdot 4 \cdot 15 = 256 - 240 = 16 \), hence \( x_{1,2} = \frac{-(-16) \pm \sqrt{16}}{2(4)} = \frac{16 \pm 4}{8} = \left\{ \frac{5}{2}, \frac{3}{2} \right\} \)

\( x \)-intercepts: \(\left( \frac{5}{3}, 0 \right); \left( \frac{3}{2}, 0 \right)\)

[c] \( x_v = \frac{-(-16)}{2(4)} = 2 \), \( y_v = 4(2)^2 - 16(2) + 15 = -1 \) (or \( y_v = \frac{-\Delta}{4(4)} = \frac{-16}{16} = -1 \))

The line is parallel to \( y = 4x \), so it must be \( y = 4(x - 2) - 1 = 4x - 9 \)

Problem 4: 5 points

Given a function: \( f(x) = \frac{(3x^2 - 6)(x - 3)}{(x^2 - 9)(x - 2)} \). Compute the following limits:

[a] (1 point) \( \lim_{x \to \infty} f(x) \)  [b] (1 point) \( \lim_{x \to 0} f(x) \)  [c] (3 points) \( \lim_{x \to 3} f(x) \)

Solution:

[a] \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3x^2(x)}{x^2(x)} = 3 \)

[b] \( \lim_{x \to 0} f(x) = \frac{(-6)(-3)}{(-9)(-2)} = 1 \)

[c] \( \lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{(3x^2 - 6)(x - 3)}{(x - 3)(x + 3)(x - 2)} = \lim_{x \to 3} \frac{3x^2 - 6}{x + 3}(x - 2) = \frac{3(3)^2 - 6}{6} = \frac{21}{6} = \frac{7}{2} \)

Problem 5: 10 points

Use Least Square method to find the equation of the line that best fits the following data:
\((1, 1); (2, 4); (3, 5); (4, 9); (5, 11)\)

Solution: \( \hat{x} = 3, \hat{y} = 6, m = 5/2, b = -3/2, y = 2.5x - 1.5 \)

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Problem 6: 5 points
Compute the derivative of the following function:

\[ f(x) = \frac{x^3 + 1}{x} + 5x \]

Solution:

\[ f(x) = x^2 + x^{-1} + 5x \] therefore \( f'(x) = 2x - x^{-2} + 5 \)

Bonus question: 5 points
Find the equation of the line which is tangent to the graph of \( y = f(x) \) at \( x = 1 \)

Solution:

\( x_0 = 1, f(x_0) = f(1) = 1 + 5 = 7, f'(x_0) = f'(1) = 2 - 5 = 6 \)
Equation of tangent line: \( y = 6(x - 1) + 7 = 6x + 1 \)

Problem 7: 5 points
Find the derivative of:

\[ f(x) = (x^3 - 9x + 1)(5x^4 + 3x - 2) \]

Solution:

\[ f'(x) = (3x^2 - 9)(5x^4 + 3x - 2) + (x^3 - 9x + 1)(20x^3 + 3) \]

Extra credit question: 5 points:
Approximate: \( Q = \frac{1}{1.002^{10}} \). Show details.

Solution:

\( f(x) = x^{-10}, x = 1, h = 0.002, f(1) = 1 \), also \( f'(x) = -10x^{-11}, f'(1) = -10 \)
Therefore, \( Q = f(x + h) \approx f(x) + hf'(x) = 1 + 0.002(-10) = 1 - 0.02 = 0.98 \)